SPECTRAL RISK MEASURES FOR CREDIT PORTFOLIOS

CLAUDIO ALBANESE AND STEPHAN LAWI

ABSTRACT. In this article, we experiment with several different risk measures such as standard deviation, value-at-risk, expected shortfall and power-law spectral measures. We consider several families of test-portfolios, one with a typical market risk profit-and-loss profile, and the others containing defaultable bonds of various credit ratings and various degree of diversification. We find that the risk measures are roughly equivalent on the market risk portfolios but differ significantly on the credit ones. In fact, value-at-risk as well as some coherent risk measures including expected shortfall have counter-intuitive features as a function of the degree of diversification for return distributions deviating substantially from the normal. Certain spectral risk measures possess the most intuitive properties regarding diversification.

1. INTRODUCTION

The quest for good risk measures that capture the risk exposure of a portfolio in terms of a single number, has led to the consideration of standard deviation [7] and value-at-risk [8]. More recently, coherent risk measures have been introduced [1, 2] and examples such as the expected shortfall [3] and spectral risk measures [4] have been considered. This article is a re-write of a 1997 working paper [5] in the light of the recent findings in [4].

Value-at-risk (*VaR*) is the most commonly used measure of market risk and is recommended by the Basel committee [6] and broadly endorsed by regulatory agencies [8]. If Π is a random variable of zero mean modeling portfolio returns within a given time horizon, the *VaR* corresponding to the percentile level $\alpha \in (0,1)$ is defined as follows:

$$VaR^{\alpha}(\Pi) = -\inf\left\{x \in \mathbb{R} : \text{Prob}[\Pi \leq x] > \alpha\right\} \tag{1}$$

Whenever return distributions are normal, *VaR* is proportional to the standard deviation of returns and is a fully satisfactory risk measure. However, if return distributions are not normal, *VaR* does not enjoy all the properties that a financially meaningful risk measure ought to have. In [2], Artzner et al. propose a set of axioms to characterize a class of financially meaningful risk measures, called *coherent*. In short and postponing more accurate definitions to the following, a risk measure $\rho(\Pi)$ is coherent if it satisfies the following property of sub-additivity:

$$\rho(\Pi_1 + \Pi_2) \leq \rho(\Pi_1) + \rho(\Pi_2) \tag{2}$$

for any two portfolios with returns modeled by the random variables Π_1 and Π_2. Said otherwise, according to a coherent risk measure, the risk of a large portfolio is less than the sum of the risk of its parts taken separately. As observed in [2], the value-at-risk measure $VaR^{\alpha}(\Pi)$ does not satisfy this property in the case of a general return distribution. *VaR* is a coherent risk measure in the particular case when risk factors have a distribution which is unimodal, the normal and Student-t distributions being examples. Discrete distributions as the ones arising in the credit risk examples in this paper, show however that *VaR* is not a coherent risk measure, in general.

Examples of coherent risk measures are given by the expected shortfall *EVaR* defined as follows

$$EVaR^{\alpha}(\Pi) = -\,N\ \mathrm{E}\left[\ \Pi\ \middle|\ \Pi \leq -\,VaR^{\alpha}(\Pi)\ \right], \tag{3}$$

where N is a normalization constant and by the spectral measure from Acerbi [4] defined as follows:

$$M_{\phi_\alpha}(\Pi) = -\int_0^1 \phi_\alpha(p) F_{\Pi}^{\leftarrow}(p)\, dp \tag{4}$$

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where $\phi_\alpha(p)$ is a decreasing function on the interval [0,1] called the risk aversion function and $F_\Pi^{\leftarrow}(\alpha) = \inf\{x \in \mathbb{R} : \mathrm{Prob}[\Pi \leq x] > \alpha\}$ is the generalized inverse probability density function. In this paper, we restrict $\phi_\alpha(p)$ to a polynomial form and define the spectral measure *SVaR* as follows:

$$SVaR_d^\alpha(\Pi) = -\,N \int_0^\alpha p^{\frac{1}{1-d}} F_\Pi^{\leftarrow}(p)dp \tag{5}$$

where N is a normalization constant such that *SVaR* equals *VaR* on normal distributions.

Coherence is more than a technical problem, but has major regulatory repercussions. Lack of coherence deprives *VaR* of part of its intuitive appeal and diminishes its reliability as a benchmark to determine capital reserves: in fact, if *VaR* is used to monitor credit risk, one can fictitiously lower the apparent credit exposure of a portfolio and the capital requirements by artificially splitting the holdings into sub-portfolios.

Due to the central limit theorem, a perfectly diversified portfolio has normal returns. Significant deviations from normality occur often and reflect imperfect diversification strategies and *diversification risk*. Market risk factors such as equity indices, foreign exchange rates, commodity prices and interest rates are continuously distributed but are not normal due to skewness and excess kurtosis. However, applications of *VaR* to market risk seldom incur in this pathology as return distributions are typically unimodal and *VaR* is coherent on those. Credit rating migrations are discrete and represent a more dramatic deviation from normality where *VaR* fails to satisfy diversification properties in a more apparent way. Events covered by insurance or re-insurance policies such as death or natural catastrophes are yet another example. Equity portfolios with positions in common stock and stock options are sometimes mapped into portfolios with aggregate variables such as market indices defined by means of an Arbitrage Pricing Theory (APT) analysis; the APT residuals are typically neglected but are usually large and if portfolios are not well diversified, can give rise to sizeable effects. If portfolios are not well diversified, deviations from normality should be carefully assessed in a quantitative manner and reflect in a higher level of capital reserves. If not quantified and monitored, diversification risk can lead to inadequate reserve capital allocations and unexpected losses.

Coherence (or lack thereof) is an issue of tangible importance in the case of portfolios exposed to credit or insurance risk, where return distributions typically deviate substantially from the normal due to the large losses associated to low-probability events. A simple example can illustrate the problem. Consider a bond, which pays $100 in one year, except if the issuer defaults, in which case it is worthless. If the probability of default is 0.11 % and credit risk is measured with respect to the $100 benchmark, then the *VaR* with 99% confidence is zero. However, the *VaR* with 99% confidence of a portfolio with 10 similar bonds issued by independent counter-parties with the same credit rating, is $100, because the event of one counter-party defaulting occurs with probability larger than 1%.

The impact of rare events on the risk profile is often compounded by finite market liquidity. Equity or bonds issued by individual corporations are typically far less liquid than future contracts on equity indices and on Treasury bonds. As a result, company specific residual risk, which is uncorrelated to market indices or moves in the term structure of interest rates, is hard to hedge. Although we do not address the specifics of the liquidity problem, price uncertainties due to finite liquidity can be incorporated into our model by adjusting correlations and the size of the residual risk factors.

In this article we construct test-portfolios by *replicating* elementary portfolios. The replica of a random variable x is a random variable y independent of x but with an identical probability distribution. Replicating the residual risk component for a certain exposure amounts to adding to the portfolio a different exposure with an identical risk profile but driven by an independent risk factor. The sum of two normal deviates is still normal, while the distribution of the sum of replicas of any other random variable approaches the normal distribution only in the limit of an infinite number of replicas. By replicating positions, we are able to quantitatively measure the deviations from normality and the degree of diversification.

We argue that coherence is just but one property that one should consider and that there are other desirable properties that are worthwhile discussing. We consider three properties: (i) positivity of marginal risk, (ii) monotonicity of specific risk and (iii) monotonicity of a measure for diversification risk.

The positivity property has the following meaning: if Π_1 and Π_2 are independent random variables describing two return distributions of zero mean, then

$$\rho(\Pi_1) \le \rho(\Pi_1 + \Pi_2). \tag{6}$$

In other words, adding an independent exposure of zero mean should increase the risk. If risk capitals are allocated according to a risk measure not satisfying this property, then one can paradoxically lower the perceived risk exposure by taking an additional independent risk. In this paper we show that assuming coherence is not sufficient to ensure that this property holds. Not even *EVaR* , which is the most popular risk measure, satisfies this property, as our examples show. Spectral measures do not satisfy it either in general, but we find that there are special choices of the spectral weight function $\phi_\alpha(p)$ which appear to greatly reduce the violations.

The specific risk property means the following: Let Π_n be the portfolio made up of n independent replicas of the portfolio Π_1. Then the specific risk per replica is a monotonously decreasing function of n, i.e.:

$$\frac{\rho(\Pi_n)}{n} \le \frac{\rho(\Pi_m)}{m} \tag{7}$$

for $1 \le m \le n$.

The monotonicity property is based on the notion of *relative diversification risk*, defined as the ratio between the risk of a portfolio and the risk one would compute if all random variables were approximated by normal deviates with the same mean and variance. The property states that the relative diversification risk of the portfolio Π_n is lower than the relative diversification risk of the portfolio Π_m whenever $1 \le m \le n$.

The paper is organized as follows: In Section 2 we introduce the notions of replicas and test-portfolios in the particular context of bonds subject to credit risk. In Section 3, we define the coherent risk measures and the diversification properties mentioned above. To illustrate the concepts behind our mathematical constructions and derivations, we specialize in section 4 to concrete numerical examples taken from the *Credit Metrics* document [9]. We present numerical evidence indicating that most coherent risk measures do not satisfy the diversification properties. Section 5 contains concluding remarks.

2. TEST-PORTFOLIOS WITH MARKET RISK AND ENTITY SPECIFIC RISK

The instruments which make part of a portfolio are typically paper such as stocks, bonds, options, etc., written by either a sovereign or a corporate entity, or by a financial intermediary who then references such an entity. The value of portfolio holdings is exposed to the risk of fluctuation for the prevailing market prices for the most traded assets, equity indices, spread curves, etc. Portfolio values are also affected by entity specific information such as credit migrations, defaults and entity specific risk. From the risk measurement viewpoint, market risk typically gives rise to unimodal risk profiles, while the return distributions arising from entity specific risk typically have a much more complex structure. As a consequence, as we discuss below, defining appropriate risk measures that allow one to assess the risk by means of a single number, is much more challenging in the case of entity specific risk.

Credit risk is one of the several instances of entity specific risk. Consider a zero coupon bond maturing at time T and let Q denote the amount of the single cash flow associated with it. If the issuer remains solvent till maturity, the payoff Q at time T will be one dollar. If the issuer defaults, Q will be a fraction, and will be paid out at the time of default, possibly earlier than T. At any time previous time t, a company is characterized by its credit rating $c(t)$. Using the Standard and Poor classification, $c(t)$ is a discrete variable that takes values in the set (AAA, AA, A, BBB, BB, B, CCC, Default). The risk neutral value of the zero coupon bond is

$$B(t,c) = E_t\left[\exp\left(-\int_s^\tau r(s,c(s))ds\right)Q\right], \tag{8}$$

where τ is the stopping time at which either the bond expires or the issuer defaults. To assess credit risk with time horizon s of cash flows occurring at a later time T, we can model the following forward prices as random variables:

$$F_{t,s,T}(c_1) = E_t \left[\exp\left(-\int_s^\tau r(s, c(s)) ds \right) Q \ , \ \ c(s) = s_1 \right]. \tag{9}$$

The gain due to a credit rating migration from C_0 to C_1 is defined as

$$M_{t,s,T}(C_0, C_1) = F_{t,s}(C_1) - \overline{F}_{t,s}(C_0), \tag{10}$$

where $\bar{F}_{t,s}(C_0)$ is the current forward price of a bond with current rating C_0, i.e.

$$\overline{F}_{t,s}(C_0) = \sum_{C_1} P(C_0 \to C_1) F_{t,s}(C_1). \tag{11}$$

This definition is such that

$$E_t \left[M_{t,s,T}(C_{0,}C_1) \right] = 0. \tag{12}$$

The matrix elements $M_{t,s,T}(C_0, C_1)$ are positive and represent gains in case the quality of the bond is upgraded, i.e. $C_1 > C_0$. Otherwise, if $C_1 < C_0$, there is a loss. The diagonal matrix elements $M_{t,s,T}(C_0, C_0)$ vanish.

In a value-at-risk calculation, t represents the current time, s is the time horizon for value-at-risk and T is one of the vertices to which cash flows are mapped. The return of the ith bond over a period $R_i(t, t{+}1)$ can be decomposed into various factors. When the factors are properly chosen, the residual returns are uncorrelated with the risk factors and among themselves. Formally,

$$r_i(t, t+1) = \sum_\alpha \beta_{\alpha,i} f_\alpha + \varepsilon_i, \tag{13}$$

where f is the vector of market (or common) risk factors and the coefficients β_α are the corresponding betas. The residual risk factors ϵ_i have zero mean and are uncorrelated with f_α. Further, for two different bonds i and j, the residuals ϵ_i and ϵ_j are uncorrelated. The risk factors f_α model the change in the fundamentals of the economy. Although their effect on different bonds are different, as reflected in the difference in $\beta_{\alpha j}$, any change in the factors f_α would affect the value of all the bonds.

Since default events occur to individual corporate or sovereign entities, they are more ”bond specific”. Although the economic environment does have an effect on all bonds' default and recovery rates, a large portion of the default risk cannot be attributed to any common risk factor. Rather, it arises from the idiosyncrasy of that specific entity, and is reflected in the specific risk factor ϵ_i. Since bond value changes associated with credit rating migrations are discrete, the specific risk factor is assumed to follow a discrete distribution. Our study focuses on the aggregation of specific risk. For a portfolio with a holdings vector $\overrightarrow{h}$, the portfolio's return $r_h(t, t + 1)$ can be expressed as:

$$r_h(t, t + 1) = \sum_i h_i r_i(t, t + 1) = \sum_{i,\alpha} h_i(\beta_{\alpha,i} f_\alpha + \epsilon_i). \tag{14}$$

Therefore, the portfolio's rate of return is attributed to the common factors f_α and the weighted sum of the specific risk for different bonds. According to the central limit theorem, the sum of residuals

$$\sum_{i,\alpha} h_i \epsilon_i \tag{15}$$

approximates a normal distribution of zero mean in the limit as the number of bonds is large and at the condition that the components h_i of the holding vector are all equal. The result extends to the case in which, loosely speaking, a large number of components h_i have roughly the same size and dominate all other components. However, if the dominant components in h are few, substantial deviations from normality occur. The goal of this article is to find financially meaningful measures to quantify the discrepancy between the sum of residuals and the normal distribution with the same mean and variance.

The numerical example we consider in this paper is given in Table 2. To facilitate comparisons, this table is taken from the popular *Credit Metrics* document [9]. Transition probabilities among credit ratings are given in Table 3.

In order to build test-portfolios with credit risk exposure, we consider financial models with a single period. The securities in the economy are bonds B, characterized by an initial credit rating $R_0(B)$ and a notional $N(B)$. The final credit rating $R_1(B)$ is a random variable. We assume that each bond B corresponds to a different residual risk factor and that residual risk factors are mutually independent random variables. We call such bonds, *credit securities*. The *replica* of a credit security B is a credit security B' with the same notional as B, i.e. $N(B') = N(B)$, for which the random variable $R(B')$ has the same distribution $R(B)$ but is uncorrelated with $R(B)$.

A *credit portfolio* is a portfolio of credit bonds whose payoff is described by a random variable Π. The payoff of the credit security B is the random variable

$$\Pi_1 = N(B) \cdot D(R_0(B), R_1(B)), \tag{16}$$

where $D(R_0, R_1)$ is a fixed payoff matrix. A credit portfolio composed of n replicas of B $(B_1, B_2, \ldots, B_n)$ has a payoff described by the random variable

$$\Pi_n = \sum_{i=1}^{n} N(B) \cdot D(R_0(B), R_1(B_i)), \tag{17}$$

where the $R_1(B_i)$ are independent random variables of same distribution.

3. Properties of Risk Measures

A risk measure is a positive mapping on the set of portfolio return distributions Π modeled as real valued random deviates of zero mean. To be financially meaningful, a risk measure must satisfy the following properties for all random payoffs Π:

Property 3.1. General Properties for Risk Measures.

- $\rho(\Pi) \geq 0$
- $\rho(\Pi) \leq |\Pi_-|_\infty$

A more questionable property which is fulfilled by the measures we consider in this article:

Property 3.2. Numeraire Invariance For all $\lambda > 0$ and all payoffs of zero mean Π, we have that

$$\rho(\lambda\Pi) \;=\; \lambda\rho(\Pi). \tag{18}$$

If a risk measure shows this property, price indeterminacy related to liquidity risk ought to be incorporated by scaling the risk factors appropriately with the size of the position.

Property 3.3. Irrelevance of Gains If Π is a random variable of zero mean and Π_- is its negative part, then $\rho(\Pi) = \rho(\Pi_-)$.

A risk measure fully appropriate for the case in which the underlying risk factors are normally distributed is the following:

Definition 3.4. The *VaR* with confidence level $u = 1 - \alpha$ for the portfolio whose profits-and-losses (P&L) are described by a random variable Π is defined as follows:

$$VaR^{\alpha}(\Pi) = -\inf\left\{x \in \mathbb{R} : \text{Prob}\left[\Pi \leq x\right] > \alpha\right\}. \tag{19}$$

The intuitive notion that a more diversified portfolio is less risky than a less diversified one can be expressed in several ways. Coherent risk measures in [2] satisfy the following additional diversification property, according to which the risk of a portfolio is less then the sum of the risk of its parts:

Property 3.5. Coherence. For all pairs of random variables of zero mean, Π_1 and Π_2, we have that

$$\rho(\Pi_1 + \Pi_2) \leq \rho(\Pi_1) + \rho(\Pi_2). \tag{20}$$

The question we address in this paper is whether the coherence property provides a sufficient characterization of all reasonable financial risk measures. We find that there are several alternative properties which are also quite reasonable but which are not shared by all coherent risk measures. Namely, consider the following properties:

Property 3.6. Positivity of Marginal Risk. If Π_1 and Π_2 are two independent random variables describing two return distributions of zero mean, then we have that

$$\rho(\Pi_1) \leq \rho(\Pi_1 + \Pi_2) \tag{21}$$

Also interesting is the following stronger property accounting for the specific risk:

Property 3.7. Monotonicity of Specific Risk. Let Π_n be the portfolio made up of n independent replicas of a portfolio Π_1. Then the specific risk per replica is a monotonously decreasing function of n, i.e. for $1 \leq m \leq n$, we have that

$$\frac{\rho(\Pi_n)}{n} \leq \frac{\rho(\Pi_m)}{m}. \tag{22}$$

The most elementary method to satisfy the diversification properties in the previous section is discussed in the *Credit Metrics* document and amounts to replacing the distribution of each random variable with a normal deviate with the same mean and variance. The *VaR* measured after this transformation is called in the following *Gaussian VaR* and denoted by *GVaR* . The ratio between the risk of a portfolio and *GVaR* is a natural measure of diversification risk. We hence define the relative diversification risk as follows:

Definition 3.8. Relative Diversification Risk. If ρ is a coherent measure on a portfolio Π. The relative diversification risk is defined as the ratio: $\dfrac{\rho(\Pi)}{GVaR(\Pi)}$.

The Gaussian diversification property is then formulated in terms of this notion:

Property 3.9. Gaussian Diversification. If Π_1 and Π_2 are two independent portfolios, then we have that

$$\frac{\rho(\Pi_1 + \Pi_2)}{GVaR(\Pi_1 + \Pi_2)} \;\leq\; \frac{\rho(\Pi_1)}{GVaR(\Pi_1)}. \tag{23}$$

As we find below, this property is quite difficult to achieve for risk measures other than standard deviation.

The several risk measures considered below need a fare basis for comparison. All coherent risk measures are indeed defined up to a constant factor. We introduce the following definition for the normalization conditions such that all risk measures return the same value on a normal distribution, ie. they all equal *VaR* .

Definition 3.10. Normalization Conditions. If Π_0 is a portfolio whose returns follow a normal distribution and Π is any independent portfolio, then the normalized risk measure $\tilde{\rho}$ is defined as follows:

$$\tilde{\rho}(\Pi) = \rho(\Pi)\, \frac{VaR^{\alpha}(\Pi_0)}{\rho(\Pi_0)}. \tag{24}$$

The normalization ratios $\dfrac{VaR^{1\%}(\Pi_0)}{\rho(\Pi_0)}$ for each of the measure defined below are given in Table 4.

A coherent risk measure was proposed by Artzner et al. [2] and called *tail conditional expectation*. We denote it *EVaR* and define it as follows:

Definition 3.11. If Π is a random variable for the profits and losses of a portfolio and $VaR^{\alpha}(\Pi)$ is the corresponding value-at-risk, then *EVaR* is defined as follows:

$$EVaR^{\alpha}(\Pi) = -\, N \, \mathrm{E}\left[\ \Pi \;\middle|\; \Pi \leq -VaR^{\alpha}(\Pi)\ \right], \tag{25}$$

where N is a normalization constant such that $EVaR^{\alpha}(\Pi_0) = VaR^{\alpha}(\Pi_0)$ if Π_0 is a normally distributed random variable.

This measure has several interesting properties, as it dominates *VaR* and has an intuitive meaning in that it assigns more weight to the left tail of the distribution. However, it might not be enough to fully satisfy the property of positivity of marginal risk when the return distribution differs from the normal.

Acerbi and Tasche proved in [3], under specific conditions such as the continuity of the return distribution, the equivalence of ***EVaR*** with the expected shortfall.

$$EVaR^{\alpha}(\Pi) = N\,\frac{1}{\alpha}\int_0^{\alpha} VaR^{p}(\Pi)dp, \tag{26}$$

where N accounts for the normalization conditions. The latter relation suggests a way to improve consistency with respect to the property of positivity of marginal risk. By altering the measure proportionally for small percentiles, we can increase the weight of the left tail of the distribution. In [4], Acerbi argues that the expected shortfall is one measure out of a large class of coherent risk measures. Following this idea, we define a spectral measure, that we call ***SVaR*** , with polynomial weight function $\phi_{\alpha}(p)$.

Definition 3.12. Let $VaR^{\alpha}(\Pi)$ be the value-at-risk for given portfolio Π and percentile α. Then the spectral risk measure, ***SVaR*** , is defined as follows:

$$SVaR_d^{\alpha}(\Pi) = N\,\frac{d-2}{d-1}\,\alpha^{-\frac{d-2}{d-1}}\int_0^{\alpha} p^{\frac{1}{1-d}}\,VaR^{p}(\Pi)dp, \tag{27}$$

where N is a normalization constant such that $SVaR_d^{\alpha}(\Pi_0) = VaR^{\alpha}(\Pi_0)$ if Π_0 is normally distributed. $d \geq 2$ is a parameter that we name *degree of freedom*.

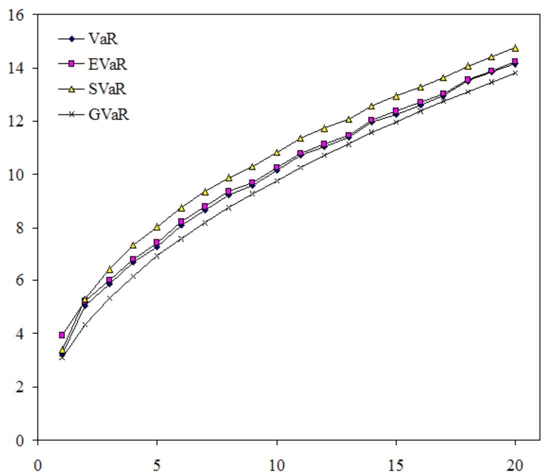
Notice that the parameter d denotes the smallest degree of freedom of a Student-t distribution with finite risk under the measure $SVaR_d^{\alpha}$.

4. Discussion of Test-Portfolios

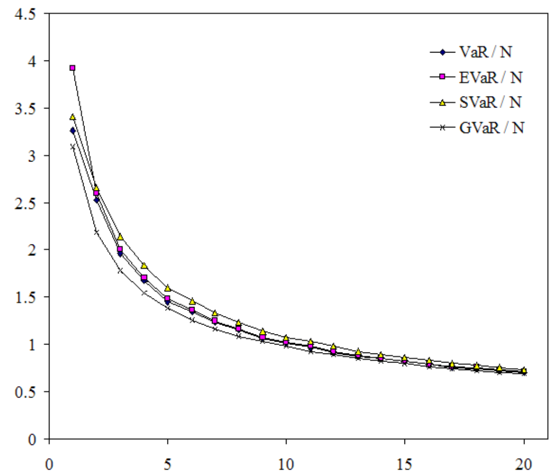
Numerical examples for the four coherent risk measures ***VaR*** , ***EVaR*** , ***SVaR*** and ***GVaR*** are displayed below in this section for a number of test portfolios. The degree of freedom d of ***SVaR*** in Definition 3.12 is set to 2.3 and the percentile level α is chosen to be 1% for all measures.

Figure 1 considers an initial P&L distribution randomly generated by 500 Student-t deviates with 4 degrees of freedom. The portfolios are then constructed recurrently by taking convolutions of the initial P&L distribution, hence considering the sum of independent replicas of these random return distributions. Figure 1(a) on the next page compares the four measures with respect to the number of replicas in the portfolio. Although the measures roughly coincide, the plots emphasize the dominance of the spectral measure ***SVaR*** over the other measures and the validity of property 3.6 of positivity of marginal risk. Figure 1(b) accounts for the monotonicity of specific risk (property 3.7), showing convex and monotonously decreasing curves. Figure 1(c) illustrates the relation of ***VaR*** , ***EVaR*** and ***SVaR*** with the Gaussian ***VaR*** and underlines the convergence of these measures to ***GVaR*** as the number of convolutions becomes large. The graph displays non-monotonous curves, unable to verify property 3.9 of Gaussian diversification. Figure 1(d) shows the shapes of the P&L distribution of the portfolio when composed of 50 replicas. The discrete shape of the P&L distribution is apparently responsible for the failure of the Gaussian diversification property. ***VaR*** , ***EVaR*** and ***SVaR*** are indeed computed over a discrete P&L histogram and increase by steps related to the width of the bin, whereas ***GVaR*** is proportional to the square root of the number of replicas.

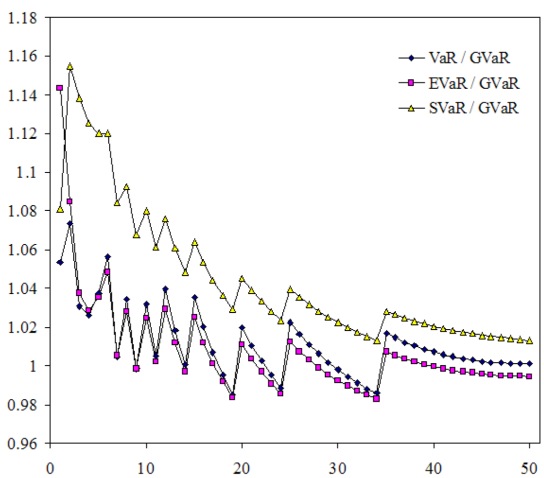
Except for Gaussian diversification, these measures capture most of the intuitive properties defined in section 3. However, one can wonder how stable the measures are under statistical estimations. Table 1 contains the results of a simulation, whereby the risk of a portfolio with just one asset is evaluated 1000 times, and at each run the risk is computed by means of a simulation in which 500 scenarios are generated. The standard deviation provides a way to rank the stability of risk measures with respect to historical estimations: we see that ***GVaR*** is the most stable measure, followed by ***VaR*** , ***EVaR*** and finally ***SVaR*** . Hence, we conclude that of the measures which account for fat-tailed distributions, ***VaR*** appears to be the most stable in the case of unimodal profit-and-loss distributions typical of market risk profiles.



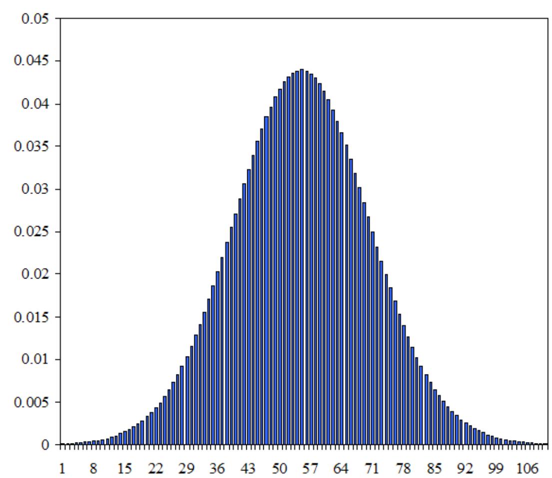
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π



(c) Gaussian diversification per number of replicas in Π



(d) Profit & Loss for 50 replicas in Π

FIGURE 1. Portfolios Π composed of N replicas of an initial P&L distribution generated by 500 Student-t deviates of 4 degrees of freedom

Measure	Average	Standard Deviation
VaR	3.75	0.50
EVaR	4.36	0.81
SVaR	4.37	1.39
GVaR	3.28	0.26

TABLE 1. Risk measures over portfolios of 500 Student-t deviates of 4 degrees of freedom (average and standard deviation over 1000 scenarios)

Figures 2 to 8 display graphs for portfolio of replicas of defaultable bonds rated AAA, AA, A, BBB, BB, B and CCC respectively. The corresponding forward prices are listed in Appendix A, Tab. 2, and are taken directly from the *Credit Metrics* technical document. The credit rating transition probabilities for each initial bond are listed in Table 3. The portfolios are built recursively over the number of replicas of the initial P&L distribution. The P&L distribution for a portfolio of N replicas is calculated by implementing the convolution of the P&L distribution of $N-1$ replicas with the P&L distribution of one replica.

Figures 2(a) to 8(a) present *VaR* , *EVaR* , *SVaR* and *GVaR* as a function of the number of replicas. Neither *VaR* nor *EVaR* is observed to enjoy the property 3.6 of positivity of marginal risk. *SVaR* on the other hand captures the latter feature, in that it is a monotonously increasing function of the degree of diversification. Deviations are observed both for portfolios made up of investement grade bonds of rating AA (Fig. 3(a)) and BBB (Fig. 5(a)), and for portfolios of junk bonds of rating single-B (Fig. 7(a)) and especially the CCC (Fig. 8(a)).

The measure *GVaR* , proportional to the square root of the number of replicas, obviously also enjoys the positivity property. By definition, *GVaR* neglects tail-risk. One observes that *GVaR* is mostly dominated by the other measures - as it should be - but there are significant exceptions to this rule. For instance, Figures 2(b) and 3(b) regarding AAA and AA bonds show that *VaR* and *EVaR* underestimate *GVaR* for portfolios with a small number of replicas. Instead, *SVaR* does dominate *GVaR* , thus recognizing the fat-tailed behaviour of these singular return distributions.

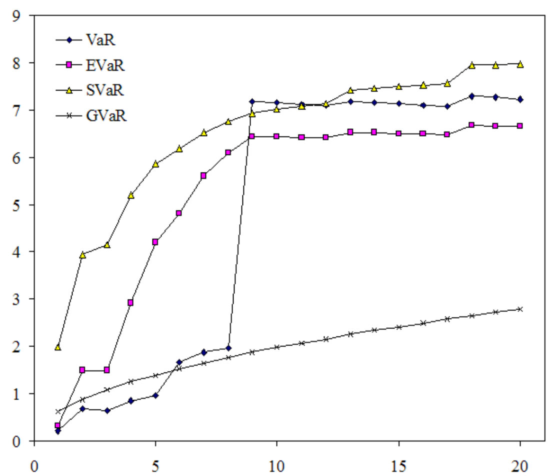
Notice that the *VaR* of AAA bonds (Fig. 2(a)) as a function of the number of replicas, achieves a maximum for 9 replicas, a behavior that contradicts all of the diversification properties in section 3. A jump in *VaR* as one goes from 16 to 17 replicas still occurs for single-A bonds (Fig. 4(a)) and disappears for single-B bonds (Fig. 7(a)). This inconvenient aspect of *VaR* is due to the lack of unimodality of the returns and the small probability of default of high rating bonds which becomes relevant only when a threshold in the number of replicas is reached. *EVaR* and *SVaR* have the merit of smoothing the shape of *VaR* , as in Figures 2(a) and 4(a) by according more weight to the left tail of the distribution.

Figures 2(b) to 8(b) investigate how the risk measures account for the specific risk per bond. For all ratings, *VaR* and *EVaR* fail to verify the second risk-diversification property: the monotonicity of specific risk (Property 3.7). This is particularly evident for AAA and AA bonds in Figures 2(b) and 3(b) where specific *VaR* and *EVaR* even underestimate specific *GVaR* for portfolios with a small number of replicas. On the other hand, the Acerbi-Tasche spectral risk measures *SVaR* in Definition 3.12 with weight $p^{\frac{d-2}{d-1}}$ display a monotonously decreasing curve for all portfolios except for CCC bonds (Fig. 8(b)), and are thus better - if not ideally - behaved. The spectral weight results in a drastic increase of *SVaR* for portfolios of a small number of replicas. The cases of single-B (Fig. 7(b)) and CCC (Fig. 8(b)) bonds are peculiar in the sense that *SVaR* underestimates *VaR* and *EVaR* in the first convolutions. The high probability of default constrains them to be all equal for small numbers of replicas and the normalization ratios account for the discrepancies pictured in the graphs.

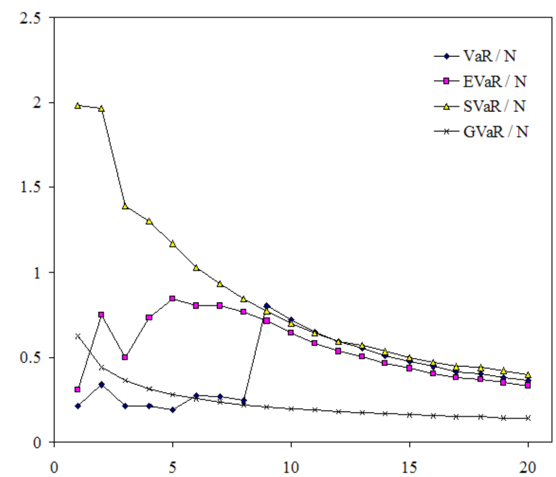
The plots in Figures 2(c) to 8(c) illustrate the relation between the risk measures and *GVaR* . The ratio tends to 1, asymptotically as the number of replicas tends to infinity, the convergence to this limit is slow, namely because of the slow convergence of the P&L distribution towards a normal distribution and its lack of unimodality (cf. Fig. 2(d)-6(d)). The fastest convergence is achieved for poorly rated bonds such as single-B (Fig. 7(c)) and CCC bonds (Fig. 8(c)) and is closely related to the shape of the P&L distribution (cf. Fig. 7(d)-8(d)). As expected, the closer the shape of the P&L distribution is to unimodal, the faster the risk measure ratio converges to 1. For high rating bonds such as the AA's, the convergence is rather slow (Fig. 3(c)) and conforms with the previous assessment, in that the low probability of default restrains the P&L distribution from a fast convergence to normal (Fig. 3(d)).

Figures 2(c) to 8(c) also show that none of the risk measures we considered satisfy the Gaussian diversification property 3.8. None of the ratios $\frac{VaR}{GVaR}$, $\frac{EVaR}{GVaR}$ or $\frac{SVaR}{GVaR}$ is a monotonously decreasing function of the number of replicas. The computation of the measures *VaR* , *EVaR* and *SVaR* over a discrete, non-unimodal P&L distribution is again responsible for the failure. The analysis of Gaussian diversification risk places a threshold on how relevant the modelling of a portfolio by a normal distribution of same mean and variance is. Portfolios of AA bonds need more than 50 replicas for the ratio $\frac{SVaR}{GVaR}$

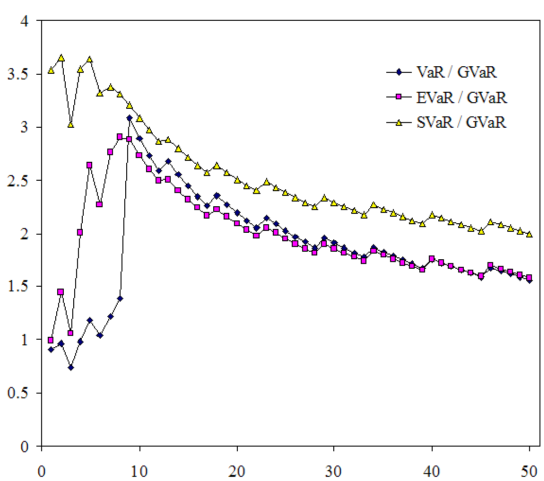
to be within 2.5 (Fig. 3(c)) - denoting a major deviation from normality - , whereas portfolios of CCC bonds need only 10 replicas for all three ratios to be within 1.1 (Fig. 8(c)). Hence, investment grade portfolios present the hardest challenges with regard to risk assessment.



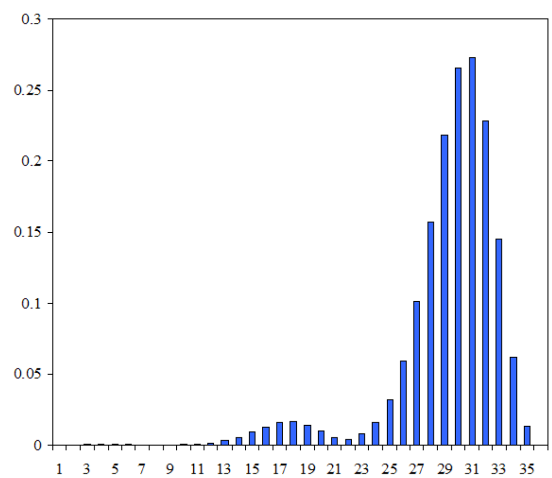
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

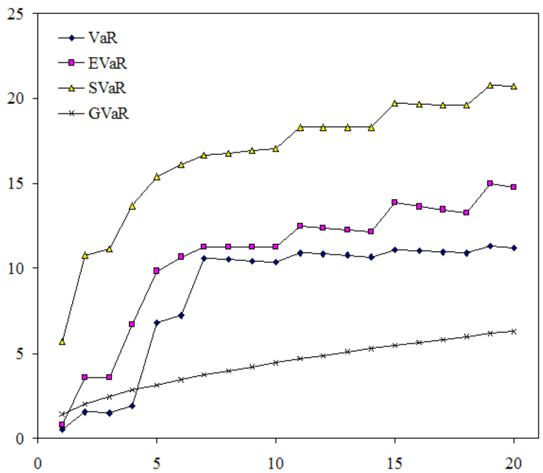


(c) Gaussian diversification

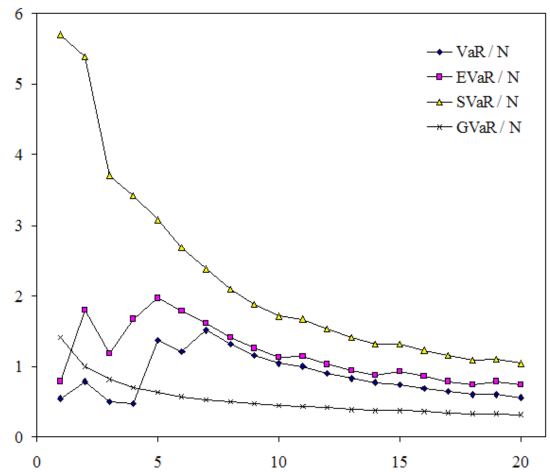


(d) Profit & Loss for 50 replicas in Π

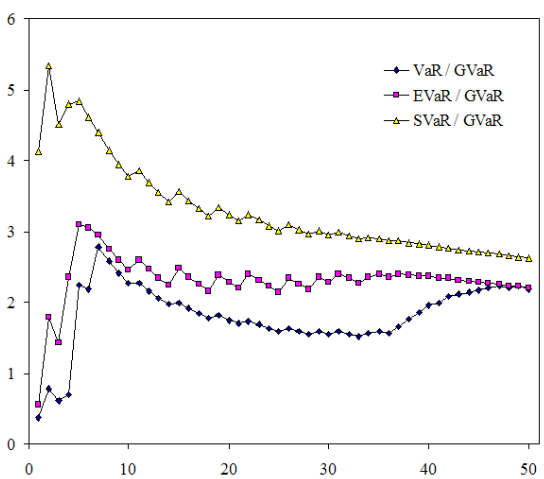
FIGURE 2. Portfolios Π composed of N replicas of a AAA Bond



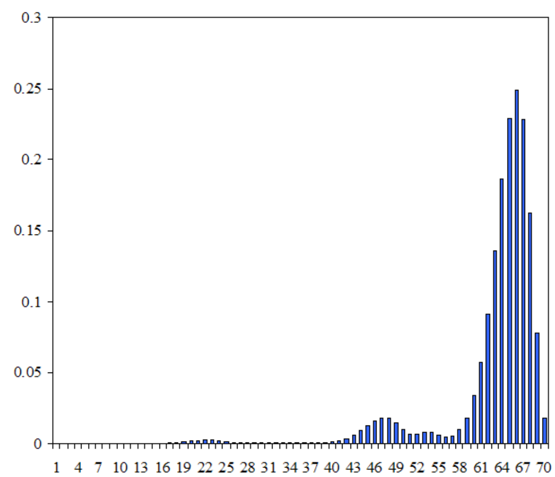
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

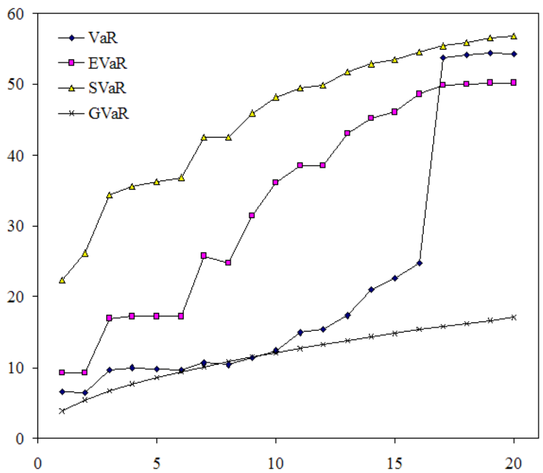


(c) Gaussian diversification

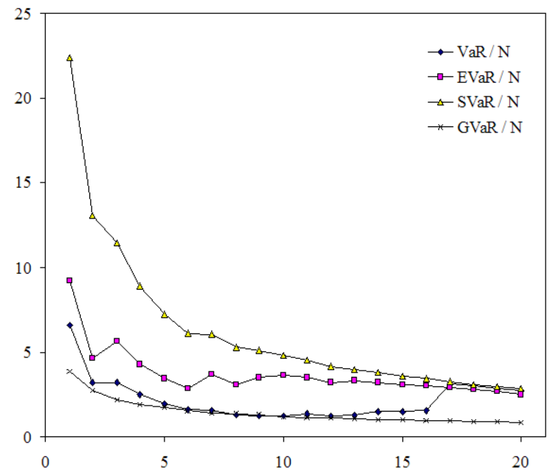


(d) Profit & Loss for 50 replicas in Π

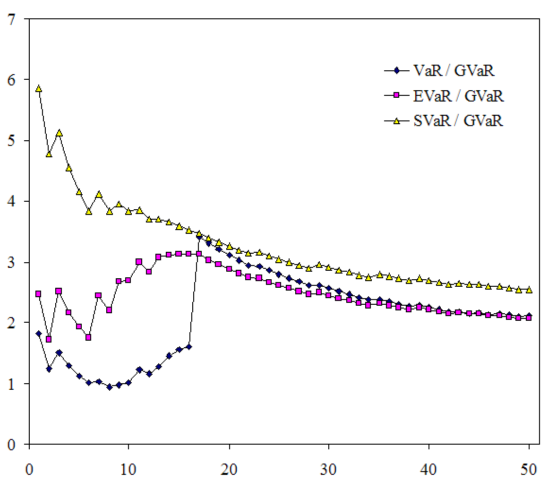
FIGURE 3. Portfolios Π composed of N replicas of a AA Bond



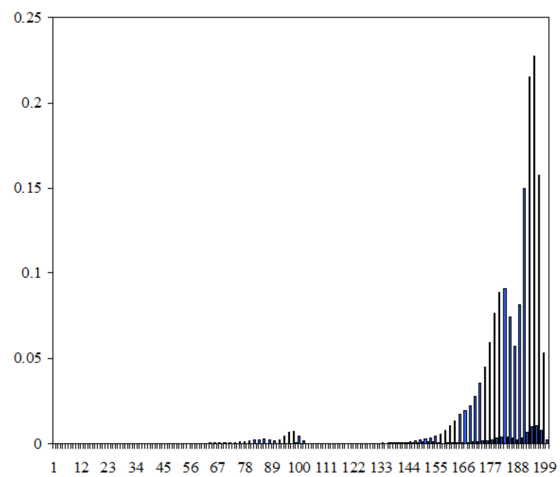
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

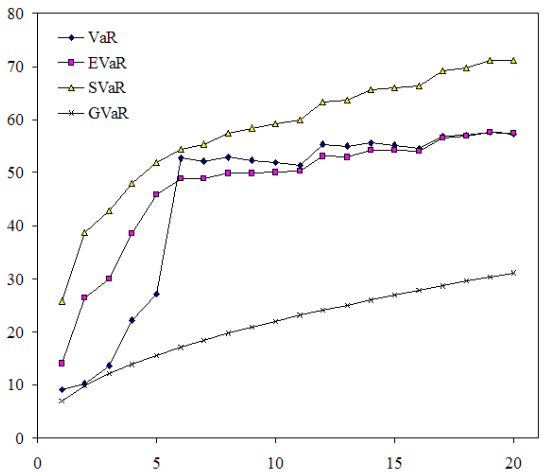


(c) Gaussian diversification

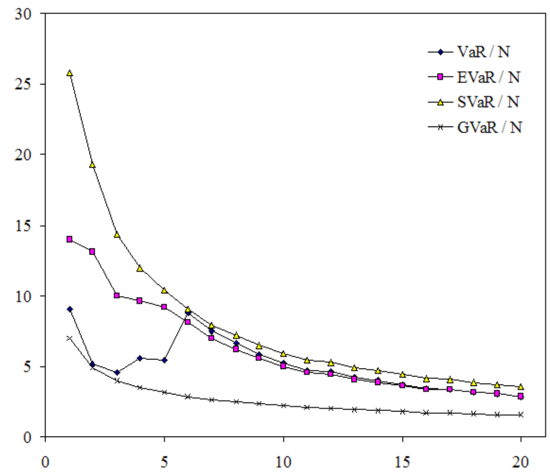


(d) Profit & Loss for 50 replicas in Π

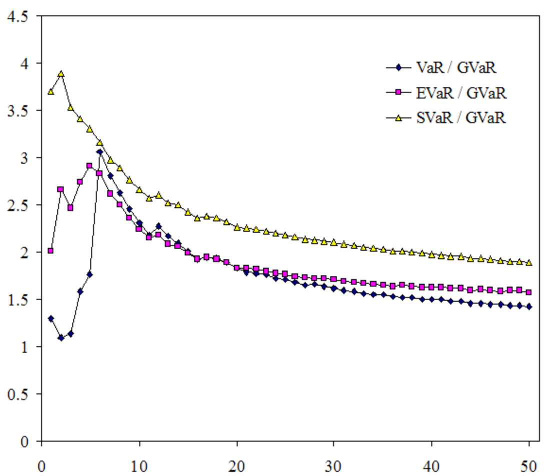
FIGURE 4. Portfolios Π composed of N replicas of a A Bond



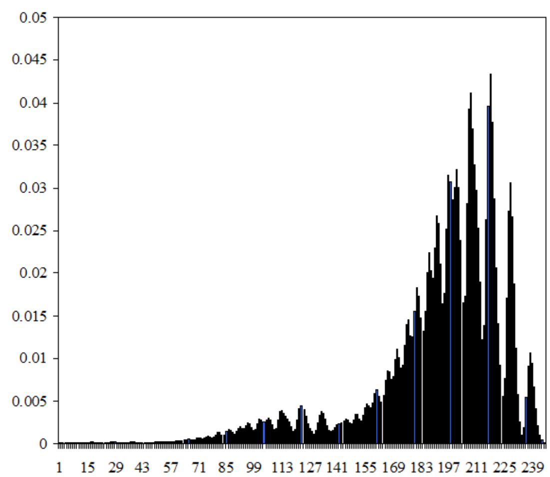
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

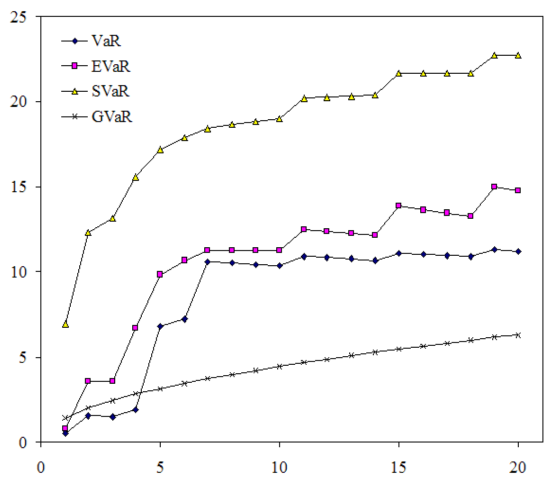


(c) Gaussian diversification

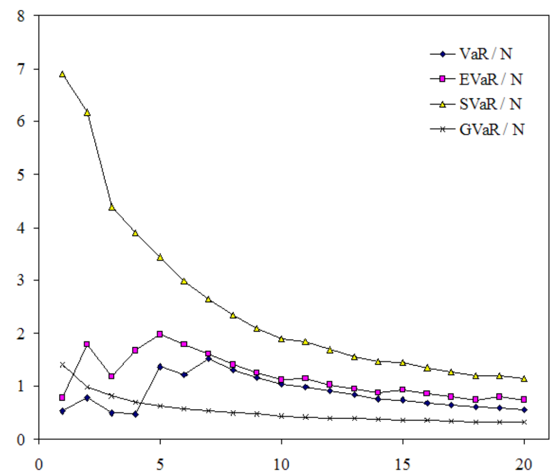


(d) Profit & Loss for 50 replicas in Π

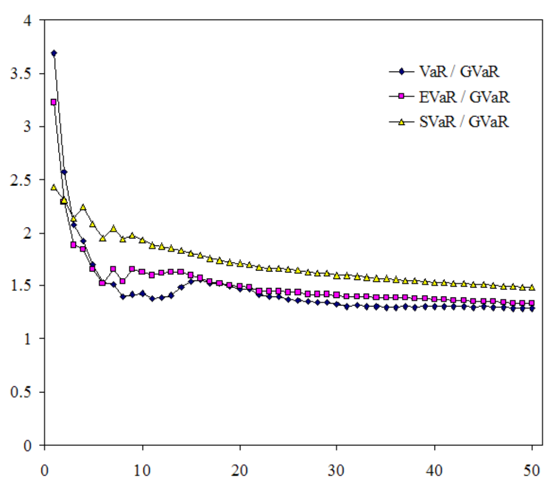
FIGURE 5. Portfolios Π composed of N replicas of a BBB Bond



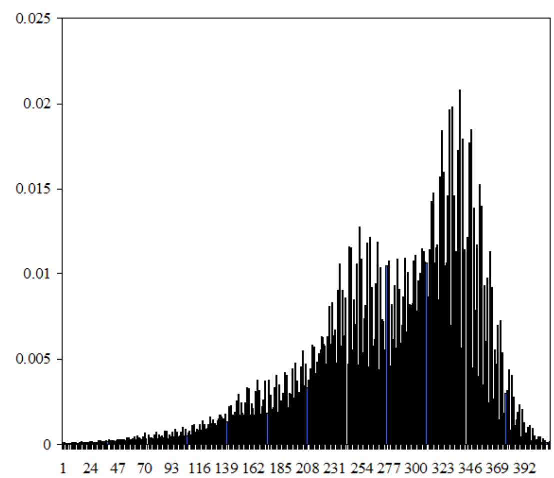
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

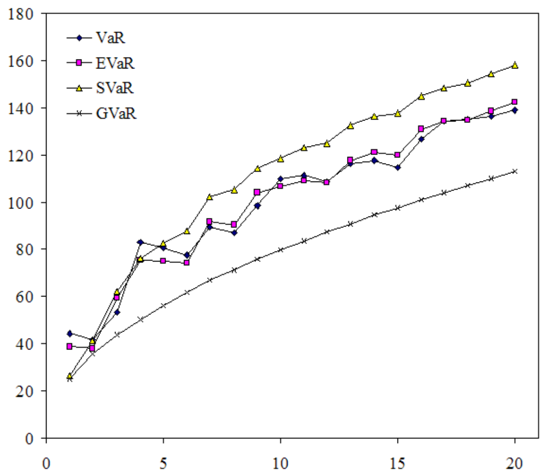


(c) Gaussian diversification

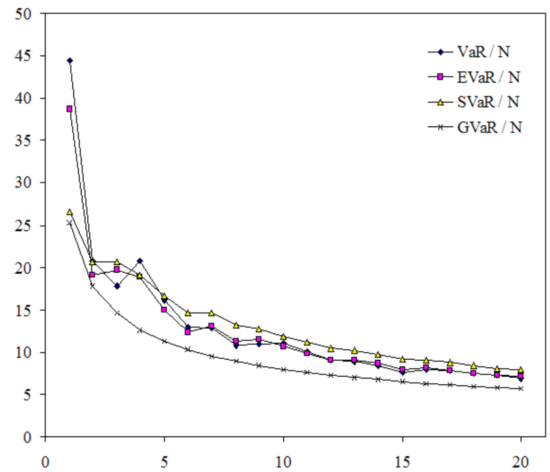


(d) Profit & Loss for 50 replicas in Π

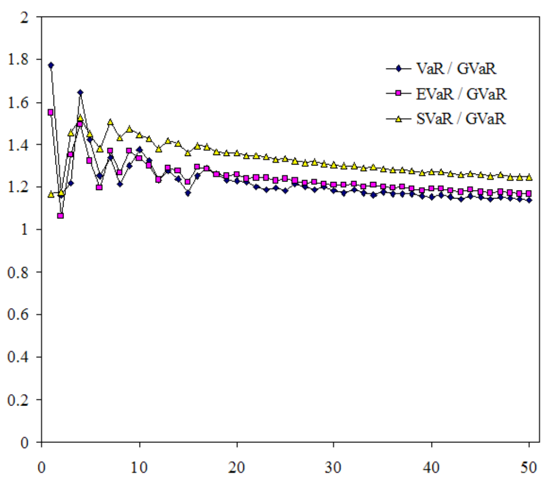
FIGURE 6. Portfolios Π composed of N replicas of a BB Bond



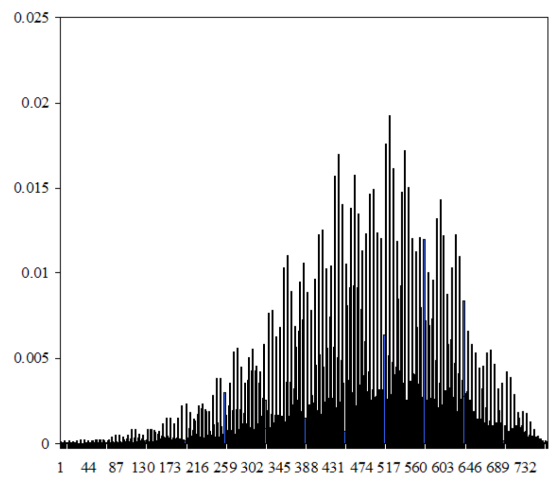
(a) Marginal risk per number of replicas in Π



(b) Specific risk per number of replicas in Π

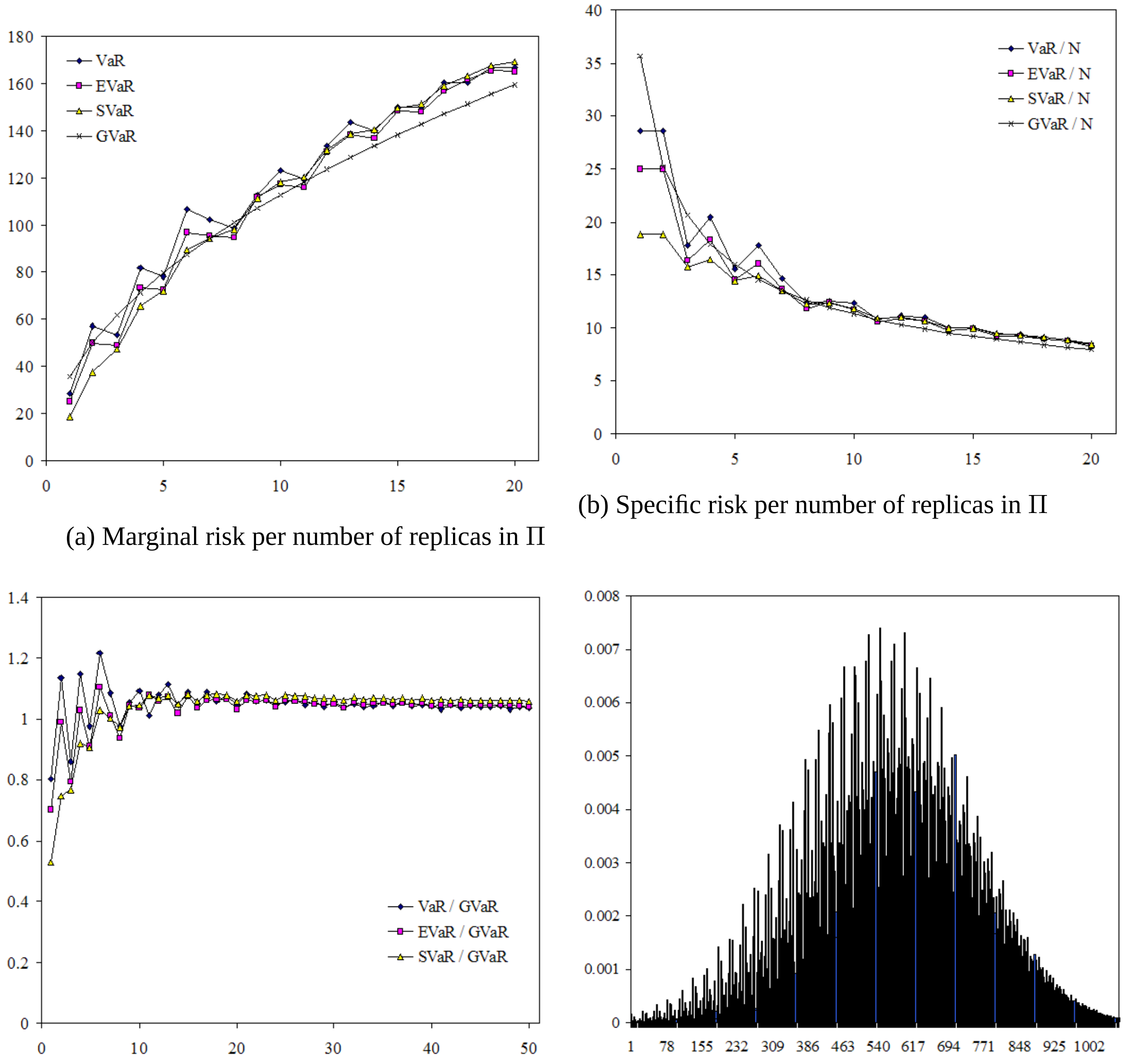


(c) Gaussian diversification



(d) Profit & Loss for 50 replicas in Π

FIGURE 7. Portfolios Π composed of N replicas of a B Bond



(a) Marginal risk per number of replicas in Π

(b) Specific risk per number of replicas in Π

(c) Gaussian diversification

(d) Profit & Loss for 50 replicas in Π

FIGURE 8. Portfolios Π composed of N replicas of a CCC Bond

5. CONCLUDING REMARKS.

Diversification risk can represent a large portion of the risk exposure. This article compares several risk measures on various families of test-portfolios, one with a typical market risk P&L profile, and the others containing defaultable bonds of various credit ratings and various degree of diversification. We find that *VaR* and *EVaR* roughly coincide with *SVaR* and dominate *GVaR* on the market risk portfolio. Among the first three measures which are sensitive to the risk of fat-tailedness of return distributions, *VaR* appears to be the most stable one with respect to historical or Montecarlo valuations. In the case of credit risk portfolios the situation is more complex as all measures we consider fail to some degree to satisfy intuitive diversification risk properties. We find that the spectral measure *SVaR* from a class recently introduced by Acerbi and Tasche is the best behaved in the group considered here and appears to enjoy the properties of positivity of marginal risk and monotonicity of specific risk.

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APPENDIX A. TABLES

Rating	Forward price
AAA	109.37
AA	109.19
A	108.66
BBB	107.55
BB	102.02
B	98.10
CCC	83.64
Default	51.13

TABLE 2. Forward Prices in one year ($)

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0.00	0.00	0.00
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0.00
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0.00	0.11	0.24	0.43	6.48	83.46	4.07	5.21
CCC	0.22	0.00	0.22	1.30	2.38	11.24	64.86	19.78

TABLE 3. Transitions Probabilities (%)

Risk Measure	Ratio
$VaR/EVaR$	0.87
$VaR/SVaR$	0.66
$VaR/GVaR$	1.00

TABLE 4. Normalization ratios with respect to *VaR*

CLAUDIO ALBANESE, DEPARTMENT OF MATHEMATICS, 100 ST. GEORGE STREET, UNIVERSITY OF TORONTO, M5S 3G3, TORONTO, CANADA
E-mail address: `albanese@math.utoronto.ca`

STEPHAN LAWI, DEPARTMENT OF MATHEMATICS, 100 ST. GEORGE STREET, UNIVERSITY OF TORONTO, M5S 3G3, TORONTO, CANADA
E-mail address: `slawi@math.utoronto.ca`