TRANSFORMATIONS OF MARKOV PROCESSES AND CLASSIFICATION SCHEME FOR SOLVABLE DRIFTLESS DIFFUSIONS

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ABSTRACT. We propose a new classification scheme for diffusion processes for which the backward Kolmogorov equation is solvable in analytically closed form by reduction to hypergeometric equations of the Gaussian or confluent type. The construction makes use of transformations of diffusion processes to eliminate the drift which combine a measure change given by Doob's h-transform and a diffeomorphism. Such transformations have the important property of preserving analytic solvability of the process: the transition probability density for the driftless process can be expressed through the transition probability density of original process. We also make use of tools from the theory of ordinary differential equations such as Liouville transformations, canonical forms and Bose invariants. Beside recognizing all analytically solvable diffusion process known in the previous literature fall into this scheme and we also discover rich new families of analytically solvable processes.

1. INTRODUCTION

An "analytically solvable" Markov process can be informally defined as follows:

Definition 1.1. A process X_t is *solvable* if its transition probability distribution can be expressed as an integral over a quadratic expression in hypergeometric functions.

This definition is very general as it includes all the well known examples in the literature such as Brownian hypergeometric Brownian motion and the Ornstein-Uhlenbeck, Bessel, square-root and Jacobi processes. It also includes a broad family of other processes which are discovered by means of the classification exercise in this paper. A similar classification problem but addressing the question of classifying all diffusion processes for which one can express the Laplace transform of the integral in analytically closed form, was addressed by C. Albanese and S. Lawi in [4].

In section 2 we briefly review the necessary definitions, facts and theorems about the diffusion processes. The most important objects which play a role in our constructions are Markov generators and transition probability densities, the speed measure and the scale and Green's functions. We also recall Feller's classification of boundary conditions for a diffusion process.

In section 3 we introduce the concept of stochastic transformations as a composition of a Doob's h-transform and a diffeomorphism, and show how to construct a complete family of stochastic transformations for a given diffusion process. We then show how these results can be generalized to arbitrary Markov process.

In section 4 we prove some useful properties of stochastic transformations and discuss the equivalence relation these transformations induce on the set of all driftless diffusions. Examples include Brownian and hypergeometric Brownian motions.

In the last section 5 we generalize the concept of stochastic transformations into that of a general transformation of the Markov generator regarded as a second order differential operator. We introduce and define "Bose invariants" which are invariant under stochastic transformations and "Liouville transformations" which act on second order differential operators while preserving the Bose invariants and thus to obtain new families of solvable processes. We conclude by stating and proving two classification theorems.

Appendix A gives necessary facts and formulas about hypergeometric functions while appendix B provides several useful facts about Ornstein-Uhlenbeck, square-root and Jacobi diffusions.

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2. Background from the theory of diffusion processes

This section is a brief introduction to the classical theory of one-dimensional diffusions: construction and probabilistic descriptions of analytical tools as the speed measure, the scale and Green function and the description of the boundary behavior of the diffusion process X_t. References on this subject include [10], [9], [12] and [7].

Let D be the (possibly infinite) interval $[D^1, D^2] \subseteq \mathbb{R}$, with $\infty \le D^1 < D^2 \le \infty$. Let X_t be a *stationary Markov process* taking values in D with transition probability function $P(t,x,A) = \mathrm{P}_x(X_t \in A)$.

Definition 2.1. The *probability semigroup* is defined as the one-parameter family of operators

$$P(t)f(x) = \int f(y)P(t,x,dy) = E_{0,x}f(X_t) \tag{2.1}$$

and the *resolvent operator* is defined as the Laplace transform of $P(t)$

$$R(\lambda)f(x) = \int_0^\infty e^{-\lambda t}P(t)f(x)dt \tag{2.2}$$

on the domain $L^\infty(D)$ of bounded measurable functions $f:\ D \to \mathbb{R}$.

We call a process X_t *conservative* if $P(t,x,D) \equiv 1$ for all t and all $x \in D$. If the process is not conservative, then we can enlarge the state space by adding a *cemetery point* Δ_∞:

$$P(t,x,\Delta_\infty) = 1 - P(t,x,D).$$

With this addition, the process X_t on the space $D \cup \Delta_\infty$ is conservative. If f is a function on D, we will extend it to $D \cup \Delta_\infty$ by letting $f(\Delta_\infty) = 0$.

Definition 2.2. The *infinitesimal generator* $\mathcal{L}$ of the process X_t is defined as follows:

$$\mathcal{L}f := \frac{d}{dt}P(0^+)f = \lim_{t\to 0^+}\frac{P(t)f - f}{t} \tag{2.3}$$

for all continuous, bounded $f:\ D \to \mathbb{R}$, such that the limit exists in the norm. The set of all these functions f is the *domain of* $\mathcal{L}$ and is denoted $\mathcal{D}(\mathcal{L})$.

Below we assume that X_t is a regular diffusion process, specified by its Markov generator

$$\mathcal{L}f = \frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) - c(x)f(x) \tag{2.4}$$

where the functions $b(x)$,$c(x)$ and $\sigma(x)$ are smooth and $c(x) \ge 0$, $\sigma(x) > 0$ in the interior of D.

Every diffusion process has three basic characteristics: its *speed measure* $m(dx)$, its *scale function* $s(x)$ and its *killing measure* $k(dx)$. For the diffusion specified by the generator (2.4) these characteristics are defined as follows:

Definition 2.3. Speed measure and killing measure are absolutely continuous with respect to the Lebesgue measure (in the interior of domain D)

$$m(dx) = m(x)dx, \quad k(dx) = k(x)dx,$$

and the functions $m(x)$, $k(x)$ and $s(x)$ are defined as follows:

$$m(x) = 2\sigma^{-2}(x)e^{B(x)}, \quad s'(x) = e^{-B(x)}, \quad k(x) = c(x)m(x) = 2c(x)\sigma^{-2}(x)e^{B(x)} \tag{2.5}$$

where $B(x) := \int^x 2\sigma^{-2}(y)b(y)dy$.

Remark 2.4. We denote by $m(x)$ a density $\frac{m(dx)}{dx}$. The same applies to the killing measure $k(dx)$.

The functions m, s and k have the following probabilistic interpretations:

- Assume $k \equiv 0$. Let $H_z := \inf\{t : X_t = z\}$ and $(a,b) \subset D$. Then

  $$\mathrm{P}_x(H_a < H_b) = \frac{s(b) - s(x)}{s(b) - s(a)}.$$

  We say that X_t is in *natural scale* if $s(x) = x$. In this case (if the process is conservative) X_t is a local martingale.

- The speed measure is characterized by the property according to which for every $t > 0$ and $x \in D$, the transition function $P(t, x, dy)$ is absolutely continuous with respect to $m(dy)$:
$$P(t, x, A) = \int_A p(t, x, y) m(dy)$$
and the density $p(t, x, y)$ is positive, jointly continuous in all variables and symmetric: $p(t, x, y) = p(t, y, x)$. Notice that the transition probability density $p(t, x, y)$ is the kernel of the operator $P(t)$ with respect to the measure $m(dy)$.
- The killing measure is associated to the distribution of the location of the process at its lifetime $\zeta := \inf\{t : X_t \notin D\}$:
$$\mathrm{P}_x(X_{\zeta-} \in A | \zeta < t) = \int_0^t ds \int_A p(s, x, y) k(dy).$$

From this point onwards we assume that there is no killing in the interior of domain D, or namely that $c(x) \equiv 0$.

Remark 2.5. Notice that the scale function can be characterized as a solution to equation
$$\mathcal{L}s(x) = 0,$$
and $s'(x)$ is proportional to the Wronskian $W_{\varphi_1,\varphi_2}(x)$, where φ_1, φ_2 are any two linearly independent solutions to
$$\mathcal{L}\varphi = \lambda\varphi.$$

Let τ be the stopping time with respect to the filtration $\{\mathcal{F}_t\}$. The process $X_{\tau\wedge t}$ is called *the process stopped at* τ and is denoted by X_t^τ.

The following lemma is required in the next sections:

Lemma 2.6. *Let $T = \inf\{t \geq 0 : \ X_t \notin \mathrm{int}(D)\}$ be the first time the process X_t hits the boundary of D. Then for each $x \in D$, $Y_t^T \equiv s(X_t^T)$ is a continuous P_x-local martingale.*

We refer to [12], vol. II, p.276 for the proof of this lemma .

The speed measure and the scale function are defined in terms of the coefficients of the generator $\mathcal{L}$. Also the inverse of the above claim holds true: if the generator $\mathcal{L}$ of the process X_t can be expressed as

$$\mathcal{L}f = \frac{d}{m(dx)} \frac{df(x)}{ds(x)} = D_m D_s f, \tag{2.6}$$

then the speed measure and the scale function define the generator of the process X_t (and thus determine the behavior of X_t up to the first time it hits the boundary of the interval). The boundary behavior of the process X_t is described by the following classical result (see [9],[10]):

Lemma 2.7. Feller classification of boundary points. *Let $d \in (D^1, D^2)$. Define functions $R(x) = m((d, x))s'(x)$ and $Q(x) = s(x)m(x)$. Fix small $\epsilon > 0$ (such that $D^1 + \epsilon \in D$). Then the endpoint D^1 is said to be:*

$$\begin{cases} \textit{regular if} & Q \in L^1(D^1, D^1 + \epsilon), \quad R \in L^1(D^1, D^1 + \epsilon) \\ \textit{exit if} & Q \notin L^1(D^1, D^1 + \epsilon), \quad R \in L^1(D^1, D^1 + \epsilon) \\ \textit{entrance if} & Q \in L^1(D^1, D^1 + \epsilon), \quad R \notin L^1(D^1, D^1 + \epsilon) \\ \textit{natural if} & Q \notin L^1(D^1, D^1 + \epsilon), \quad R \notin L^1(D^1, D^1 + \epsilon) \end{cases} \tag{2.7}$$

The same holds true for D^2.

Next we elaborate on the probabilistic meaning of different types of boundaries.

Regular or exit boundaries are called *accessible*, while entrance and natural boundaries are called *inaccessible*.

An *exit* boundary can be reached from any interior point of D with positive probability. However it is not possible to start the process from an exit boundary.

The process cannot reach an *entrance* boundary from any interior point of D, but it is possible to start the process at an entrance boundary.

A *natural* boundary cannot be reached in finite time and it is impossible to start a process from the natural boundary. The natural boundary D_1 is called *attractive* if $X_t \to D^1$ as $t \to \infty$.

A *regular* boundary is also called *non-singular*. A diffusion reaches a non-singular boundary with positive probability. In this case the characteristics of the process do not determine the process uniquely and one has to specify boundary conditions at each non-singular boundary point: if $m(\{D^i\}) < \infty$, $k(\{D^i\}) < \infty$, then the boundary conditions are

$$
\begin{cases} g(D^1)m(\{D^1\}) - \frac{df(D^1)}{ds(x)} + f(D^1)k(\{D^1\}) = 0, \\ g(D^2)m(\{D^2\}) + \frac{df(D^2)}{ds(x)} + f(D^2)k(\{D^2\}) = 0. \end{cases} \tag{2.8}
$$

where $g := \mathcal{L}f$ for $f \in \mathcal{D}(\mathcal{L})$.

The following terminology is used: the left endpoint D^1 is called

- *reflecting*, if $m(\{D^1\}) = k(\{D^1\}) = 0$,
- *sticky*, if $m(\{D^1\}) > 0$, $k(\{D^1\}) = 0$,
- *elastic*, if $m(\{D^1\}) = 0$, $k(\{D^1\}) > 0$.

A diffusion process X spends no time and does not die at a reflecting boundary point. X does not die, but spends a positive amount of time at a sticky point (which in the case $m(\{D^1\}) = \infty$ is called an *absorbing boundary* - the process stays at D^1 forever after hitting it). X does not spend any time at elastic boundary - it is either reflected or dies with positive probability after hitting D^1 (in the limit $k(\{D^1\}) = \infty$ we call D^1 a *killing boundary*, since that X is killed immediately if it hits D^1).

Let the interval D be an infinite interval, for example of the form $[D^1, \infty)$. We say that the process X_t *explodes* if the boundary $D^2 = \infty$ is an accessible boundary. Using the previous lemma one can see that the process explodes if and only if for some $\epsilon > 0$

$$
R(x) = m((D^1 + \epsilon, x))s'(x) \in L^1(D^1 + \epsilon, \infty). \tag{2.9}
$$

In section 3 below, we construct two linearly independent solutions to the ODE

$$
\mathcal{L}\varphi(x) = \lambda\varphi(x), \quad \lambda > 0, \quad x \in D. \tag{2.10}
$$

The probabilistic description of these solutions is given by the following lemma (see [12], vol. II, p. 292):

Lemma 2.8. *For $\lambda > 0$ there exist an increasing $\varphi_\lambda^+(x)$ and a decreasing $\varphi_\lambda^-(x)$ solutions to equation (2.10). These solutions are convex, finite in the interior of the domain D and are related to the Laplace transform of the first hitting time H_z as follows:*

$$
E_x\left(e^{-\lambda H_z}\right) = \begin{cases} \frac{\varphi_\lambda^+(x)}{\varphi_\lambda^+(z)}, & x \le z, \\ \frac{\varphi_\lambda^-(x)}{\varphi_\lambda^-(z)}, & x \ge z. \end{cases} \tag{2.11}
$$

The functions $\varphi_\lambda^+(x)$ and $\varphi_\lambda^-(x)$ are also called the *fundamental solutions* of equation (2.10). These functions are linearly independent and their Wronskian can be computed as follows:

$$
W_{\varphi_\lambda^+,\varphi_\lambda^-}(x) = \frac{d\varphi_\lambda^+(x)}{dx}\varphi_\lambda^-(x) - \varphi_\lambda^+(x)\frac{d\varphi_\lambda^-(x)}{dx} = w_\lambda s'(x), \tag{2.12}
$$

thus the Wronskian with respect to $D_s = d/ds(x)$ is constant:

$$
W_{\varphi_\lambda^+,\varphi_\lambda^-}(x) = \frac{d\varphi_\lambda^+(x)}{ds(x)}\varphi_\lambda^-(x) - \varphi_\lambda^+(x)\frac{d\varphi_\lambda^-(x)}{ds(x)} = w_\lambda. \tag{2.13}
$$

The following theorem due to W. Feller characterizes boundaries in terms of solutions to the equation (2.10) and will be used in section 3:

Theorem 2.9. (i) *The boundary point D^2 is regular if and only if there exist two positive, decreasing solutions φ_1 and φ_2 of (2.10) satisfying*

$$
\lim_{x\to D^2} \varphi_1(x) = 0, \quad \lim_{x\to D^2} \frac{d\varphi_1(x)}{ds(x)} = -1, \quad \lim_{x\to D^2} \varphi_2(x) = 1, \quad \lim_{x\to D^2} \frac{d\varphi_2(x)}{ds(x)} = 0. \tag{2.14}
$$

(ii) *The boundary point D^2 is exit if and only if every solution of (2.10) is bounded and every positive decreasing solution φ_1 satisfies*

$$\lim_{x \to D^2} \varphi_1(x) = 0, \quad \lim_{x \to D^2} \frac{d\varphi_1(x)}{ds(x)} \le 0. \tag{2.15}$$

(iii) *The boundary point D^2 is entrance if and only if there exists a positive decreasing solution φ_1 of (2.10) satisfying*

$$\lim_{x \to D^2} \varphi_1(x) = 1, \quad \lim_{x \to D^2} \frac{d\varphi_1(x)}{ds(x)} = 0, \tag{2.16}$$

and every solution of (2.10) independent of φ_1 is unbounded at D^2. In this case no nonzero solution tends to 0 as $x \to D^2$.

(iv) *The boundary point D^2 is natural if and only if there exists a positive decreasing solution φ_1 of (2.10) satisfying*

$$\lim_{x \to D^2} \varphi_1(x) = 0, \quad \lim_{x \to D^2} \frac{d\varphi_1(x)}{ds(x)} = 0, \tag{2.17}$$

and every solution of (2.10) independent of φ_1 is unbounded at D^2.

In cases (i) and (ii), all solutions of (2.10) are bounded near D_2 and there is a positive increasing solution z such that $\lim_{x \to D^2} z(x) = 1$. In cases (iii) and (iv) every positive, increasing solution z satisfies $\lim_{x \to D^2} z(x) = \infty$.

Another important characteristic of a Markov process is the *Green function*:

Definition 2.10. The *Green function* $G(\lambda, x, y)$ is defined as the Laplace transform of $p(t, x, y)$ in time variable:

$$G(\lambda, x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y) dt. \tag{2.18}$$

The Green function is symmetric and it is the kernel of the resolvent operator $R(\lambda) = (\mathcal{L} - \lambda)^{-1}$ with respect to $m(dx)$.

The Green function can be conveniently expressed in terms of functions $\varphi^+_\lambda(x)$ and $\varphi^-_\lambda(x)$ as:

$$G(\lambda, x, y) = \begin{cases} w_\lambda^{-1} \varphi^+_\lambda(x) \varphi^-_\lambda(y), & x \le y \\ w_\lambda^{-1} \varphi^+_\lambda(y) \varphi^-_\lambda(x), & y \le x. \end{cases} \tag{2.19}$$

A diffusion X_t is said to be *recurrent* if $\mathrm{P}_x(H_y < \infty) = 1$ for all $x, y \in D$. A diffusion which is not recurrent is called *transient*. A recurrent diffusion is called *null recurrent* if $E_x(H_y) = \infty$ for all $x, y \in D$ and *positively recurrent* if $E_x(H_y) < \infty$ for all $x, y \in D$. The following is a list of useful facts concerning recurrence, Green function and speed measure:

- X_t is recurrent if and only if $\lim_{\lambda \searrow 0} G(\lambda, x, y) = \infty$.
- X_t is transient if and only if $\lim_{\lambda \searrow 0} G(\lambda, x, y) < \infty$.
- X_t is positively recurrent if and only if $m(D) < \infty$. In the recurrent case we have: $\lim_{\lambda \searrow 0} \lambda G(\lambda, x, y) = \frac{1}{m(D)}$ and the speed measure $m(dx)$ is a *stationary (invariant) measure* of X_t:

$$mP(t)(A) := \int_A m(dx) P(t, x, A) = m(A).$$

3. STOCHASTIC TRANSFORMATIONS

Let's ask a question: how can we transform a Markov diffusion process X_t in such a way that the transition probability density of the transformed process can be expressed through the probability density of X_t? First of all let's take a look at what transformations are available.

Given a stochastic process (X_t, P) one has the following obvious choice:

- *change of variables* (change of state space for the process):
$$X_t = (X_t, \mathrm{P}) \mapsto Y_t = (Y(X_t), \mathrm{P}),$$
where $Y(x)$ is a diffeomorphism $Y : D_x \to D_y$.
- *change of measure*:
$$(X_t, \mathrm{P}) \mapsto (X_t, \mathrm{Q}),$$
where measure Q is absolutely continuous with respect to P: $d\mathrm{Q}_t = Z_t d\mathrm{P}_t$.
- *time change*:
$$X_t = (X_t, \mathcal{F}_t, \mathrm{P}) \mapsto \tilde{X}_t = (X_{\tau_t}, \mathcal{F}_{\tau_t}, \mathrm{P}),$$
where τ_t is an increasing stochastic process, $\tau_0 = 0$.
- and at last we can combine the above transformations in any order.

In this section we will discuss only the first two types of transformations. The stochastic time change can be efficiently used to add jumps or "stochastic volatility" to the processes and is important for applications to Mathematical Finance (see [3],[2]).

The next obvious question is: do these transformations preserve the solvability of the process? The answer is always "yes" for the change of variables transformation (we assume that the function $Y(x)$ is invertible): the probability density of the process $Y_t = Y(X_t)$ is given by:

$$p_Y(t, y_0, y_1) = p_X(t, X(y_0), X(y_1)), \quad y_i \in D_y,$$

where $X = X(y) = Y^{-1}(y)$ and the speed measure of Y_t is

$$m_Y(dy) = m_Y(y)dy = m_X(X(y))X'(y)dy.$$

What can we say about the measure change transformation? We want the new measure Q to be absolutely continuous with respect to P, thus there exists a nonnegative process Z_t, such that

$$\frac{d\mathrm{Q}_t}{d\mathrm{P}_t} = Z_t.$$

The process Z_t must be a (local) martingale. Under what conditions on Z_t does this transformation preserves solvability?

Informally speaking, the probability density of the process $X_t^{\mathrm{Q}} = (X_t, \mathrm{Q})$ is given by

$$p_{X^{\mathrm{Q}}}(t, x_0, x_1) m_{X^{\mathrm{Q}}}(x_1) = E^{\mathrm{Q}}(\delta(X_{s+t}^{\mathrm{Q}} - x_1) | X_s = x_0),$$

where we used the fact that the transformed process is stationary. Using a formula of change of measure under conditional expectation we obtain

$$p_{X^{\mathrm{Q}}}(t, x_0, x_1) m_{X^{\mathrm{Q}}}(x_1) = \frac{1}{Z_s} E^{\mathrm{P}}(Z_{s+t} \delta(X_{s+t}^{\mathrm{Q}} - x_1) | X_s = x_0).$$

Since we want the transformed process to be Markov, we see that this can be the case if and only if Z_s depends only on the value of X_s, thus the process Z_t can be represented as

$$Z_t = h(X_t, t),$$

for some positive function $h(x, t)$.

The next step is to note that the dynamics of X_t under the new measure is:

$$dX_t = \left(b(X_t) + \sigma^2(X_t) \frac{h_x(X_t, t)}{h(X_t, t)} \right) dt + \sigma(X_t) dW_t^{\mathrm{Q}}.$$

Now we see that if we want the transformed process to be stationary, h_x/h must be independent of t, which gives us the following expression for the function $h(x,t)$:

$$h(x,t) = h(x)g(t).$$

Since we don't want to introduce killing in the interior of the domain D, the process $Z_t = h(X_t)g(t)$ must be a martingale (at least a local one), thus we have the following equation

$$\frac{dg(t)}{dt}h(x) + g(t)\mathcal{L}_X h(x) = 0.$$

After separating the variables we find that there exists a constant ρ such that $g(t) = e^{-\rho t}$, and the function $h(x)$ is a solution to the following eigenfunction equation:

$$\mathcal{L}_X h(x) = \rho h(x). \tag{3.1}$$

This discussion leads us to our main definition:

Definition 3.1. The *stochastic transformation* is a triple

$$\{\rho, h, Y\} \tag{3.2}$$

where ρ and $h(x)$ define the absolutely continuous measure change through the formula

$$d\mathrm{P}^h_t = \exp(-\rho t)h(X_t)d\mathrm{P}_t, \tag{3.3}$$

and $Y(x)$ is a diffeomorphism $Y : D_x \to D_y \subseteq \mathbb{R}$, such that the process $(Y_t, \mathrm{P}^h) = (Y(X_t), \mathrm{P}^h)$ is a driftless process.

Remark 3.2. We deliberately do not require Y_t to be a (local) martingale, since as we will see later Y_t is not conservative in general. Though using lemma (2.6) we see that Y_t is a driftless process if and only if Y_t stopped at the boundary of D_y is a (local) martingale.

3.1. Main Theorem.

The following theorem gives an explicit way to find all stochastic transformations defined in (3.1):

Theorem 3.3. *Let X_t be a stationary Markov diffusion process under the measure P on the domain $D_x \subseteq \mathbb{R}$ and admitting a Markov generator $\mathcal{L}_X$ of the form*

$$\mathcal{L}_X f(x) = b(x)\frac{df(x)}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2 f(x)}{dx^2}. \tag{3.4}$$

Assume $\rho \geq 0$.

Then $\{\rho, h, Y\}$ is a stochastic transformation if and only if

$$\begin{cases} h(x) = c_1\varphi^+_\rho(x) + c_2\varphi^-_\rho(x), \\ Y(x) = \frac{1}{h(x)}(c_3\varphi^+_\rho(x) + c_4\varphi^-_\rho(x)), \end{cases} \tag{3.5}$$

where $c_i \in \mathbb{R}$ are parameters ,$c_1, c_2 \geq 0$, $c_1c_4 - c_2c_3 \neq 0$, and $\varphi^+_\rho(x)$ and $\varphi^-_\rho(x)$ are increasing and decreasing solutions, respectively, to the differential equation

$$\mathcal{L}_X\varphi(x) = \rho\varphi(x).$$

Before we give a proof of this theorem we need to present some theory.

3.2. Doob's h-transform.

Definition 3.4. A positive function $h(x)$ is called ρ-*excessive* for the process X if the following two statements hold true:

(i) $e^{-\rho t}E(h(X_t)|X_0 = x) \leq h(x)$
(ii) $\lim_{t\to 0} E(h(X_t)|X_0 = x) = h(x)$.

An ρ-excessive function h is called ρ-*invariant* if for all $x \in D_x$ and $t \geq 0$

$$e^{-\rho t}E(h(X_t)|X_0 = x) = h(x).$$

Remark 3.5. Function $h(x)$ is ρ-excessive (ρ-invariant) if and only if the process $\exp(-\rho t)h(X_t)$ is a positive supermartingale (martingale). Thus, if a function $h(x)$ is zero at some x_0, then $h \equiv 0$ in D_x.

One can check that for every $x_1 \in D_x$ functions $\varphi_\rho^+(x)$, $\varphi_\rho^-(x)$ and $x \to G_X(\rho, x, x_1)$ are ρ-excessive. As the following lemma shows, these functions are *minimal* in the sense that any other excessive function (except the trivial example $h \equiv C$) can be expressed as a linear combination of them (see [7]):

Lemma 3.6. *Let $h(x)$ be a positive function on D_x such that $h(x_0) = 1$. Then h is ρ-excessive if and only if there exists a probability measure ν on $D_x = [D^1, D^2]$, such that for all $x \in D_x$*

$$\begin{aligned} h(x) &= \int_{[D^1,D^2]} \frac{G_X(\rho,x,x_1)}{G_X(\rho,x_0,x_1)} \nu(dx_1) = \\ &= \int_{(D^1,D^2)} \frac{G_X(\rho,x,x_1)}{G_X(\rho,x_0,x_1)} \nu(dx_1) + \frac{\varphi_\rho^-(x)}{\varphi_\rho^-(x_0)} \nu(\{D^1\}) + \frac{\varphi_\rho^+(x)}{\varphi_\rho^+(x_0)} \nu(\{D^2\}). \end{aligned}$$

Measure ν is called the representing measure of h.

Definition 3.7. The coordinate process X_t under the measure P^h defined by

$$\mathrm{P}^h(A|X_0 = x) = E\left(e^{-\rho t} \frac{h(X_t)}{h(x)} \mathbb{I}_A | X_0 = x\right), \tag{3.6}$$

is called *Doob's h-transform* or *ρ-excessive transform* of X. We will denote this process by (X, P^h) (or in short X^h).

The following lemma gives the expression of the main characteristics of the h-transform of X:

Lemma 3.8. *The process X^h is a regular diffusion process with:*

- *Generator*

$$\mathcal{L}_{X^h} = \frac{1}{h} \mathcal{L}_X h - \rho \tag{3.7}$$

with drift and diffusion terms given by

$$b_{X^h}(x) = b_X(x) + \sigma_X^2(x) \frac{h_x(x)}{h(x)}, \quad \sigma_{X^h}(x) = \sigma_X(x). \tag{3.8}$$

- *The speed measure and the scale function of the process X^h are*

$$m_{X^h}(x) = h^2(x) m_X(x), \quad s'_{X^h}(x) = h^{-2}(x) s'(x). \tag{3.9}$$

- *The transition probability density (with respect to the speed measure $m_{X^h}(dx_1)$)*

$$p_{X^h}(t, x_0, x_1) = \frac{e^{-\rho t}}{h(x_0) h(x_1)} p_X(t, x_0, x_1). \tag{3.10}$$

- *The Green function*

$$G_{X^h}(\lambda, x_0, x_1) = \frac{1}{h(x_0) h(x_1)} G_X(\rho + \lambda, x_0, x_1). \tag{3.11}$$

- *The killing measure*

$$k_{X^h}(dx) = \frac{m_X(x) \nu(dx)}{G_{X^h}(0, X_0, x)}. \tag{3.12}$$

Proof. We will briefly sketch the proof. Using the formula (3.6) we find the semigroup for the process X^h:

$$P_{X^h}(t) f(x) = e^{-\rho t} \frac{1}{h(x)} P_X(t)(hf)(x),$$

and from this formula we find the expression for the Markov generator and formulas for the drift and diffusion. Now,

$$B(x) = \int^x \frac{2 b_{X^h}(y)}{\sigma_{X^h}^2(y)} dy = \int^x \frac{2 b_X(y)}{\sigma_X^2(y)} dy + 2 \log(h(x)),$$

from which formula we compute the expressions for the speed measure, scale function, probability density and Green function.

Finally, let's prove the formula for the killing measure: the infinitesimal killing rate c_{X^h} can be computed as $\mathcal{L}_{X^h}1$, thus we find:

$$\mathcal{L}_{X^h}1 = \frac{1}{h}\mathcal{L}_X h - \rho = \frac{1}{h}\mathcal{L}_X \int_{[D^1,D^2]} \frac{G_X(\rho,x,x_1)}{G_X(\rho,X_0,x_1)}\nu(dx_1) - \rho.$$

Now assuming that $\nu(dx) = \nu(x)dx$ and using the fact that the Green function satisfies the following equation

$$\mathcal{L}_X G_X(\rho,x,x_1) = \rho G_X(\rho,x,x_1) + \delta(x_1 - x),$$

we find

$$c_{X^h}(x) = \mathcal{L}_{X^h}1 = \frac{1}{h}\int_{[D^1,D^2]} \frac{\rho G_X(\rho,x,x_1) + \delta(x_1 - x)}{G_X(\rho,X_0,x_1)}\nu(x_1)dx_1 - \rho = \frac{\nu(x)}{h(x)G_X(\rho,X_0,x)},$$

thus the killing measure can be computed as

$$k_{X^h}(dx) = c_{X^h}(x)m_{X^h}(x)dx = \frac{m_X(x)\nu(dx)}{G_{X^h}(0,X_0,x)},$$

which ends the proof. □

Remark 3.9. Note that we have a nonzero killing measure on the boundary of D_x and no killing in the interior of D_x if and only if the representing measure $\nu(dx)$ is supported on the boundary of D_x, that is

$$h(x) = \nu(\{D^1\})\varphi_\rho^-(x) + \nu(\{D^2\})\varphi_\rho^+(x) = c_1\varphi_\rho^-(x) + c_2\varphi_\rho^+(x).$$

Remark 3.10. To understand the probabilistic meaning of the Doob's h-transform it is useful to consider the procedure of constructing new process by conditioning X_t on some event (the event A can be that the process stays in some interval or that it has some particular maximum or minimum value). The mathematical description follows:

Let the probability density $p_t(x,y|A(t_0,t_1))$ be the conditional density defined as follows:

$$p_t(x,y|A(t_0,t_1))dy = \mathrm{P}(X_{t+\delta t} \in dy | X_t = x, A(t_0,t_1)).$$

Assume that the event $A(t_0,t_1)$ is $\mathcal{F}_{t_1}$-measurable and it satisfies the semigroup property: $\mathrm{P}(A(t_0,t_1)|B) = \mathrm{P}(A(t_0,t') \cap A(t',t_1)|B)$ for any event B and $t' \in (t_0,t_1)$.

Let the probability $\pi(x,t;A(t_0,t_1)$ be defined as

$$\pi(x,t;A(t_0,t_1)) = \mathrm{P}(A(t_0,t_1)|X(t) = x).$$

Then function π satisfies the following backward Kolmogorov equation:

$$\frac{\partial \pi}{\partial t} + \mathcal{L}_X\pi = \frac{\partial \pi(x,t;A(t,T))}{\partial t} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \pi(x,t;A(t,T))}{\partial x^2} + b(x)\frac{\partial \pi(x,t;A(t,T))}{\partial x} = 0.$$

The boundary conditions depend on event $A(t,T)$.

The conditioned drift $b(x;A(t,T)$ and conditioned volatility $\sigma(x;A(t,T)$ are given by

$$\begin{aligned} b(x;A(t,T)) &= b(x) + \sigma^2(x)\frac{\pi_x(x,t;A(t,T))}{\pi(x,t;A(t,T))}, \\ \sigma(x;A(t,T)) &= \sigma(x). \end{aligned} \tag{3.13}$$

Lets consider some examples of the h-transform:

(i) **Brownian bridge through h-transform** Let $X_t = W_t$ be the brownian motion. Fix some x_0 and consider the event $A(t;T) = \{X_T = x_1\}$. Then the function $\pi(x,t;A(t,T))$ is the probability density of Brownian motion:

$$\pi(x,t;A(t,T)) = p_{T-t}(x,x_1) = \frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{(x_1-x)^2}{2(T-t)}\right).$$

One can show that X^h is just the Brownian bridge - brownian motion conditioned on the event $X_T = x_1$ (note that this process is not time-homogeneous since in this case we are not using the ρ-excessive transform).

The conditional drift of the process X^h, computed by the formula (3.13) is equal to

$$b(x; A(t,T)) = \frac{\pi_x}{\pi} = \frac{x_0 - x}{T - t},$$

which is another way to prove that X^h is a Brownian bridge.

(ii) **Brownian Motion, $X_t = W_t$.** Increasing and decreasing solutions to equation

$$\mathcal{L}^X f = \frac{1}{2}\frac{d^2 f(x)}{dx^2} = \rho f(x)$$

are given by

$$\varphi_\rho^+(x) = e^{\sqrt{2\rho}x}, \qquad \varphi_\rho^-(x) = e^{-\sqrt{2\rho}x}$$

Note that in this case the process $e^{-\rho t}h(X_t)$ is a martingale. Let $h(x) = \varphi_\rho^+(x)$, then the process $X_t^h = W_t^h + \sqrt{2\rho}t$ is Brownian motion with drift. Note that since $e^{-\rho t}h(X_t)$ is a martingale, the transformed process is still conservative, but it has a completely different behavior: for example the $X_t^h \to +\infty$ as $t \to \infty$ and $D^2 = \infty$ becomes an attractive boundary. This is a typical situation for the ρ-excessive transforms: as we will see later, the following result is true: the transformed process X^h is either nonconservative with killing at the boundary, or in the case it is conservative, X_t^h converges to one of the two boundary points as $t \to \infty$.

We will return to the example of Brownian motion later in section (4.1).

(iii) Let X_t be a transient process with zero killing measure. Then $X_t \to D^1$ or $X_t \to D^2$ with probability 1 as $t \to \zeta$. Let X^+ be the $\varphi_0^+(x)$ transform of X. Note that $\varphi_0^+(x)$ is a constant multiple of $\mathrm{P}_x(X_t \to D^2 \text{ as } t \to \zeta)$, thus X^+ is identical in law to X, given that $X_t \to D^2$ as $t \to \zeta$, or otherwise X^+ has the property:

$$\mathrm{P}_x(\lim_{t\to\zeta} X_t^+ = D^2) = 1.$$

3.3. **Proof of the main theorem.** Now we are ready to give the proof of the main theorem (3.3):

Proof. Lets prove first that the process

$$Z_t = e^{-\rho t}h(X_t) = e^{-\rho t}(c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x))$$

is a positive supermartingale. By applying Ito formula we find:

$$dZ_t = \left(-\rho Z_t + e^{-\rho t}(\mathcal{L}_X h)(X_t)\right)dt + e^{-\rho t}h'(X_t)\sigma(X_t)dW_t = e^{-\rho t}h'(X_t)\sigma(X_t)dW_t,$$

since $\mathcal{L}_X h = \rho h$. Thus Z_t is a local martingale. Since it is also positive it is actually a supermartingale (by Fatou lemma). Thus

$$E(e^{-\rho t}h(X_t)|X_0 = x) \leq h(x), \tag{3.14}$$

and $\exp(-\rho t)h(X_t)$ correctly defines an absolutely continuous measure change.

Lemmas (3.6) and (3.8) show that the converse statement is also true: if function $h(x)$ can be used to define an absolutely continuous measure change, then $e^{\rho t}h(X_t)$ is a supermartingale, thus $h(x)$ is a ρ-excessive function. Since we want the transformed process X^h to have no killing in the interior of D_x, representing measure $\nu(dx)$ must be supported at the boundaries (see remark (3.9)), thus we have the representation $h(x) = c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)$.

To prove the second statement of the theorem, we note first that $Y(X_t)$ has a zero drift if and only if it is in the natural scale: thus we need to check that function $Y(x)$ given by formula (3.5) is equal to the scale function $s_{X^h}(x)$ (up to an affine transformation), which we can check by direct computation

$$Y'(x) = \frac{d}{dx}\left(\frac{\varphi(x)}{h(x)}\right) = \frac{\varphi'(x)h(x) - h'(x)\varphi(x)}{h^2(x)} = \frac{W_{\varphi,h}(x)}{h^2(x)} = s'_{X^h}(x),$$

where φ is an arbitrary solution of $\mathcal{L}_X\varphi = \rho\varphi$ linearly independent of h, (thus it can be represented as $\varphi = c_3\varphi_\rho^+ + c_4\varphi_\rho^-$ with $c_1c_4 - c_2c_3 \neq 0$). This ends the proof of the main theorem. ∎

Remark 3.11. The statement that $Y(X_t)$ is a driftless process is an analog of the following lemma:

Lemma 3.12. *Let* $\mathrm{Q} \cong \mathrm{P}$, *and* $Z_t = \frac{d\mathrm{Q}_t}{d\mathrm{P}_t}$. *An adapted cadlag process* M_t *is a* P*-local martingale if and only if* M_t/Z_t *is a* Q*-local martingale.*

Note that this lemma does not assume that the process is a diffusion process and one could use it to define stochastic transformations for arbitrary Markov processes. One could argue as follows: $e^{-\rho t}h(X_t)$ and $e^{-\rho t}(c_3\varphi_\rho^+(X_t) + c_4\varphi_\rho^-(X_t))$ are local martingales under P, thus if we define a measure change density $Z_t = e^{-\rho t}h(X_t)$ the process

$$Y(X_t) = \frac{e^{-\rho t}(c_3\varphi_\rho^+(X_t) + c_4\varphi_\rho^-(X_t))}{Z_t} = \frac{c_3\varphi_\rho^+(X_t) + c_4\varphi_\rho^-(X_t)}{h(X_t)}$$

is a Q local martingale. However, as we will see later, in some cases Z_s is not a martingale, thus the measure change is not equivalent in general and we can't use this argument.

3.4. **Generalization of stochastic transformations to arbitrary multidimensional Markov processes.** The method of constructing stochastic transformations described in theorem 3.3 can be generalized to jump processes and multidimensional Markov processes: let X_t be a stationary process on the domain $D_x \subset \mathbb{R}^d$ with Markov generator $\mathcal{L}_X$. The first step is to find two linearly independent solutions to the equation

$$\mathcal{L}_X\varphi = \rho\varphi.$$

Assume that for some choice of c_1, c_2 the function $h(x) = c_1\varphi_1(x) + c_2\varphi_2(x)$ is positive. Then one has to prove that the process

$$Z_t = e^{-\rho t}h(X_t)$$

is a local martingale (or a supermartingale), thus it can be used to define a new measure P^h by the formula (3.6). Then we define the function $Y(x) : \mathbb{R}^d \to \mathbb{R}$ as

$$Y(x) = \frac{c_3\varphi_1(x) + c_4\varphi_2(x)}{h(x)}.$$

Now one can check that the generator of X_t^h is given by

$$\mathcal{L}_{X^h} = \frac{1}{h}\mathcal{L}_X h - \rho,$$

and to prove that the (one-dimensional) process $Y(X_t)$ is driftless one could argue as follows:

$$\mathcal{L}_{X^h}Y(x) = \left(\frac{1}{h}\mathcal{L}_X h - \rho\right)\frac{\varphi}{h} = \frac{1}{h}\mathcal{L}_X\left(h\frac{\varphi}{h}\right) - \rho\frac{\varphi}{h} = \frac{1}{h}\mathcal{L}_X\varphi - \rho\frac{\varphi}{h} = 0,$$

since $\varphi(x) = c_3\varphi_1(x) + c_4\varphi_2(x)$ is also a solution to $\mathcal{L}_X\varphi = \rho\varphi$.

Note that this "proof" does not use any information about the process X_t and thus it is very general. For example in [2],[5],[6] authors use this method to construct solvable driftless processes on the lattice.

4. Properties of stochastic transformations and examples

The following lemma summarizes the main characteristics of the process Y_t:

Lemma 4.1. *The process* $Y_t = (Y(X_t), \mathrm{P}^h)$ *is a regular diffusion process with*

- *Generator*

$$\mathcal{L}_Y = \frac{1}{2}\sigma_Y^2(y)\frac{d^2}{dy^2} \tag{4.1}$$

where volatility function is given by:

$$\sigma_Y(Y(x)) = \sigma_X(x)Y'(x) = \sigma_X(x)\frac{CW(x)}{h^2(x)}. \tag{4.2}$$

($W(x) = s'_X(x)$ *is the Wronskian of* $\varphi_\rho^+, \varphi_\rho^-$*).*

- *The speed measure and the scale function of the process Y_t are*

$$m_Y(y) = \frac{1}{\sigma_Y^2(y)}, \quad s_Y(y) = y. \tag{4.3}$$

- *The transition probability density (with respect to the speed measure $m_Y(dy_1)$) is*

$$p_Y(t,y_0,y_1) = p_{X^h}(t,x_0,x_1) = \frac{e^{-\rho t}}{h(x_0)h(x_1)} p_X(t,x_0,x_1), \tag{4.4}$$

  where $y_i = Y(x_i)$.
- *The Green function*

$$G_Y(\lambda,y_0,y_1) = G_{X^h}(\lambda,x_0,x_1) = \frac{1}{h(x_0)h(x_1)} G_X(\rho+\lambda,x_0,x_1). \tag{4.5}$$

Lemma 4.2. *Let X_t be a diffusion process on $D_x = [D^1, D^2]$ with both boundaries being inaccessible. If $c_2 = 0$ $(c_1 = 0)$ the domain D_y of the process Y_t is an interval of the form $[y_0, \infty)$ or $(-\infty, y_0]$, where $y_0 = c_3/c_1$ $(y_0 = c_4/c_2)$. In the case $c_1 \neq 0$ and $c_2 \neq 0$, the domain D_y is a bounded interval of the form $[y_0, y_1] = [c_4/c_2, c_1/c_3]$ or $[y_0, y_1] = [c_1/c_3, c_4/c_2]$.*

Proof. Remember that

$$Y(x) = \frac{c_3\varphi_\rho^+(x) + c_4\varphi_\rho^-(x)}{c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)}$$

The statements of the lemma follows easily from the fact that φ_ρ^+ is a positive increasing function, finite at D^1 and infinite at D^2 and φ_ρ^- is positive decreasing function, infinite at D^1 and finite at D^2 (see 2.9). □

Lemma 4.3. *The processes X^h and Y are transient.*

Proof. This follows from the fact that

$$\lim_{\lambda \searrow 0} G_{X^h}(\lambda,x_0,x_1) = \frac{1}{h(x_0)h(x_1)} G_X(\rho,x_0,x_1) < \infty,$$

thus X_t^h is a transient process (see section 2). □

The previous lemma tells us that with probability one X_t^h (and Y_t) visits every point in its domain only a finite number of times. Thus it converges as $t \to \infty$. Since with probability one it can not converge to a point in the interior of D_x, it must converge to the point on the boundary (or to the cemetery point Δ_∞ if the process is not conservative). Actually an even stronger result can be obtained by means of martingale theory:

Lemma 4.4. *If the process Y_t is conservative, then*

$$\mathrm{P}^h(\lim_{t\to\infty} Y_t = Y_\infty) = 1. \tag{4.6}$$

and Y_∞ is integrable. The random variable Y_∞ is supported at the boundary of the interval D_y: in the case $D_y = [y_1, \infty)$ we have $Y_\infty = y_1$ a.s., while in the general case $D_y = [y_1, y_2]$ we have that Y_t converges to Y_∞ also in L_1 and the distribution of Y_∞ is

$$\mathrm{P}(Y_\infty = y_2 | Y_0 = y_0) = \frac{y_0 - y_1}{y_2 - y_1}, \quad \mathrm{P}(Y_\infty = y_1 | Y_0 = y_0) = \frac{y_2 - y_0}{y_2 - y_1}. \tag{4.7}$$

Proof. If Y_t is conservative, then Y_t is a local martingale bounded from below (or above), thus it is a supermartingale bounded from below (or a submartingale bounded from above), thus it converges as $t \to \infty$. If $D_y = [y_1, y_2]$ and the process Y_t is bounded, then it is a uniformly integrable martingale and it converges to Y_∞ also in L_1, moreover

$$Y_t = E(Y_\infty | \mathcal{F}_t),$$

thus $y_0 = EY_t = EY_\infty$, from which we can find the distribution of Y_∞. □

Lemma 4.5. *The stochastic transformation $(X, \mathrm{P}) \mapsto (Y, \mathrm{P}^h)$ given by $\{\rho, h(x), Y(x)\}$ is invertible. The inverse transformation $(Y, \mathrm{P}^h) \mapsto (X, \mathrm{P})$ is given by $\{-\rho, 1/h(x), X(y)\}$.*

Proof. If the process X_t is conservative with inaccessible boundaries, the process Y_t will be a transient driftless process (possibly with accessible boundaries). Let's check that the process

$$Z_t = e^{\rho t} \frac{1}{h(X_t^h)}$$

is a P^h martingale:

$$E_x^{\mathrm{P}^h}(Z_t) = E_x^{\mathrm{P}} \left(e^{-\rho t} \frac{h(X_t^h)}{h(x)} Z_t \right) = \frac{1}{h(x)} E_x^{\mathrm{P}} 1 = Z_0,$$

since the initial process X_t is conservative and $E_x^{\mathrm{P}} 1 = 1$. Thus the transformation $\{-\rho, 1/h(x), X(y)\}$ maps the process $Y_t = (Y(X_t), \mathrm{P}^h)$ back into the process $X_t = (X_t, \mathrm{P})$. □

Definition 4.6. We will say that (X, P) and (Y, Q) *are related by a stochastic transformation* and will denote it by writing

$$X \sim Y$$

if there exists a stochastic transformation $\{\rho, h, Y\}$ which maps $(X, \mathrm{P}) \mapsto (Y, \mathrm{P}^h)$.

Lemma 4.7. *The relation $X \sim Y$ is an equivalence relation.*

Proof. We need to check that for all X, Y and Z

(i) $X \sim X$
(ii) if $X \sim Y$, then $Y \sim X$
(iii) if $X \sim Y$ and $Y \sim Z$, then $X \sim Z$.

The first property is obvious; the second was proved in the previous lemma. To check the third, let $\{\rho_1, h_1(x), Y(x)\}$ ($\{\rho_2, h_2(y), Z(y)\}$) be the stochastic transformation relating X and Y (Y and Z). Then $\{\rho_1 + \rho_2, h_1(x)h_2(Y(x)), Z(Y(x))\}$ is a stochastic transformation mapping $X \mapsto Z$. □

Thus "$\sim$" relation divides all the Markov stationary driftless diffusions into equivalence classes, which will be denoted by

$$\mathfrak{M}(X) = \mathfrak{M}(X, \mathrm{P}) = \{(Y, \mathrm{Q}) : \quad (Y, \mathrm{Q}) \sim (X, \mathrm{P})\}. \tag{4.8}$$

Later in lemma 5.7 we give a convenient criteria to determine whether two processes X and Y are in the same equivalence class (can be mapped into one another by a stochastic transformation).

Now we illustrate with some real examples the usefulness of the concept of stochastic transformation. We will review some well known examples (geometric Brownian motion, quadratic volatility family, CEV processes) and show how these processes can be obtained by a stochastic transformation, and in the case of Ornstein-Uhlenbeck, CIR and Jacobi processes we will construct new families of solvable driftless processes and study their boundary behavior.

4.1. **Brownian Motion.** Let $X_t = W_t$ be the Brownian motion process. Then the Markov generator is

$$\mathcal{L}^X = \frac{1}{2} \frac{d^2}{dx^2}.$$

Fix $\rho > 0$. Functions φ_ρ^+ and φ_ρ^- are given by:

$$\varphi_\rho^+(x) = e^{\sqrt{2\rho} x}, \qquad \varphi_\rho^-(x) = e^{-\sqrt{2\rho} x} \tag{4.9}$$

The next step is to fix any two positive c_1, c_2 and set $h = c_1 \varphi_\rho^+ + c_2 \varphi_\rho^-$. Note that $e^{-\rho t} h(X_t)$ is a martingale (a sum of two geometric brownian motions).

We consider separately two cases - (a) one of c_1, c_2 is zero (D_y is unbounded) and (b) both c_1, c_2 are positive.

Let's assume first that $c_1 = 0$ and $c_2 = 1$, thus

$$h(x) = \varphi_\rho^-(x) = e^{-\sqrt{2\rho} x}.$$

Thus the function $Y(x)$ is

$$Y(x) = \frac{c_3 \varphi_\rho^+(x) + c_4 \varphi_\rho^-(x)}{h(x)} = c_3 e^{2\sqrt{2\rho} x} + c_4,$$

and its inverse $x = X(y)$ is

$$X(y) = \frac{1}{2\sqrt{2\rho}} \log\left(\frac{|y-c_4|}{|c_3|}\right).$$

Using formula 4.2 we find the volatility function of the process Y_t:

$$\sigma_Y(y) = \sigma_X(X(y))\frac{1}{X'(y)} = C(y-y_1), \tag{4.10}$$

and $Y_t - y_1$ is the well known geometric brownian motion.

Note that this process is a martingale, it is transient and $\lim_{t\to\infty} Y_t = y_1$ a.s.

Now let's consider the general case: $c_1 > 0$, $c_2 > 0$. Assume that $c_4/c_2 > c_3/c_1$. Then

$$Y(x) = \frac{c_3\varphi_\rho^+(x) + c_4\varphi_\rho^-(x)}{c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)} = \frac{c_3 e^{2\sqrt{2\rho}x} + c_4}{c_1 e^{2\sqrt{2\rho}x} + c_2},$$

and

$$X(y) = \frac{1}{2\sqrt{2\rho}}\left(\log\left(\frac{|c_2|}{|c_1|}\right) + \log\left(|c_4/c_2 - y|\right) - \log\left(|y - c_3/c_1|\right)\right).$$

Thus the derivative $X'(y)$ is

$$X'(y) = \frac{1}{C(c_4/c_2 - y)(y - c_3/c_1)},$$

and again using formula (4.2) we find the volatility function of the process Y_t

$$\sigma_Y(y) = C(c_4/c_2 - y)(y - c_3/c_1) = C(y_2 - y)(y - y_1), \quad y_2 > y_1. \tag{4.11}$$

and we obtain the quadratic volatility family. In this case the process Y_t is a uniformly integrable martingale. As $t \to \infty$ the process Y_t converges to the random variable Y_∞ with distribution supported on the boundaries y_1, y_2 and given by equation (4.7).

Remark 4.8. We have proved so far that starting from Brownian motion we can obtain the martingale processes with volatility function

$$\sigma_Y(y) = a_2 y^2 + a_1 y + a_0,$$

where the polynomial $a_2y^2 + a_1y + a_0$ is either linear or has two real zeros. However theorem 5.7 tells us that the process Y_t is related to W_t for all choices of coefficients a_i. Let's consider the example of the volatility function

$$\sigma_Y(y) = 1 + y^2 \tag{4.12}$$

to understand what happens in this case.

The process Y_t with volatility (4.12) is supported on the whole real line $\mathbb{R}$ and one can prove using Khasminskii's explosion test that this process explodes in finite time. We can obtain this process starting from Brownian motion by an analog of the stochastic transformation with $\rho < 0$. For example, let $\rho = -\frac{1}{2}$. Then solutions to equation

$$\mathcal{L}_X\varphi(x) = \frac{1}{2}\frac{d^2\varphi(x)}{dx^2} = \rho\varphi(x)$$

are given by functions $\sin(x)$ and $\cos(x)$. Let's choose

$$h(x) = \cos(x), \quad Y(x) = \frac{\sin(x)}{h(x)} = \tan(x).$$

Then $Y'(x) = \frac{1}{\cos^2(x)} = 1 + \tan^2(x) = 1 + y^2$, and we obtain the volatility function given in (4.12). Function $Y(x) = \tan(x)$ is infinite when $x = \pi/2 + k\pi$, $k \in \mathbb{Z}$, thus the process $Y_t = (Y(W_t), \mathrm{P}^h)$ explodes in finite time.

Thus we have proved the following

Lemma 4.9. *Starting with Brownian motion W_t we can obtain the class of quadratic volatility models:*

$$\mathfrak{M}(W_t) = \{Y_t : \sigma_Y(y) = a_2y^2 + a_1y + a_0\}. \tag{4.13}$$

4.2. **Bessel processes.** Let X_t be a Bessel process defined by generator

$$\mathcal{L}^X = a\frac{d}{dx} + \frac{1}{2}\sigma^2 x\frac{d^2}{dx^2}.$$

We assume that $\alpha = \frac{2a}{\sigma^2} - 1 > 0$ (thus the process never reaches the boundary point $x = 0$). Then we prove the following:

Lemma 4.10. *As a particular case one can cover the CEV (constant-elasticity-of-variance) models with volatility function $\sigma_Y(y) = c(y - y_0)^\theta$.*

Proof. Choose $\rho = 0$. Then the eigenvalue equation is

$$a\frac{d\varphi(x)}{dx} + \frac{1}{2}\sigma^2 x\frac{d^2\varphi(x)}{dx^2} = \rho\varphi(x) = 0.$$

The two linearly independent solutions are $\varphi_0^+(x) = 1$ and $\varphi_0^-(x) = x^{-\alpha}$. We put $c_2 = 0$, thus $h(x) = x^{-\alpha}$ and $Y(x)$ is

$$Y(x) = \frac{c_3 x^{-\alpha} + c_4}{c_1 x^{-\alpha}} = A + Bx^\alpha.$$

Thus we can express $x = X(y) = c_1(y - y_0)^{\frac{1}{\alpha}}$ and we find that

$$Y'(X(y)) = (X'(y))^{-1} = c_2(y - y_0)^{1-\frac{1}{\alpha}}.$$

Note that since $\alpha > 1$, the power $1 - \frac{1}{\alpha}$is positive. Using formula (4.2) we can compute the volatility

$$\sigma_Y(y) = \sigma\sqrt{X(y)}Y'(X(y)) = c(y - y_0)^\theta,$$

where $\theta = 1 - \frac{1}{2}\frac{1}{\alpha}$. □

Remark 4.11. Note that we were able to compute the explicit form in these cases because that we could find the inverse function $x = X(y)$ explicitly. This is not the case for most applications, though, and we often have to use numerical inversion instead.

4.3. **Ornstein-Uhlenbeck processes - OU family of martingales.** Let X_t be the Ornstein-Uhlenbeck process

$$dX_t = (a - bX_t)dt + \sigma dWt,$$

discussed in detail in section B.1.

Without loss of generality we can assume that $a = 0$ (otherwise we can consider the process $X_t - \frac{a}{b}$). Process X_t satisfies the following property, X_t has the same distribution as $-X_t$:

$$\mathrm{P}_x(X_t \in A) = \mathrm{P}_{-x}(-X_t \in A). \tag{4.14}$$

Functions $\varphi_\rho^-(x)$ and $\varphi_\rho^+(x)$ are solutions to the ODE:

$$\frac{1}{2}\sigma^2\frac{d^2\varphi(x)}{dx^2} - bx\frac{d\varphi(x)}{dx} = \rho\varphi(x). \tag{4.15}$$

One can check that the function $\varphi_\rho^-(x)$ is given by

$$\varphi_\rho^-(x) = \sqrt{\pi}\left(\frac{M(\frac{\rho}{2b}, \frac{1}{2}, \frac{b}{\sigma^2}x^2)}{\Gamma(\frac{1}{2} + \frac{\rho}{2b})} - 2\sqrt{\frac{b}{\sigma^2}}x\frac{M(\frac{\rho}{2b} + \frac{1}{2}, \frac{3}{2}, \frac{b}{\sigma^2}x^2)}{\Gamma(\frac{\rho}{2b})}\right) \tag{4.16}$$

and due to the symmetry of X_t (see (4.14)) we have $\varphi_\rho^+(x) = \varphi_\rho^-(-x)$.

Remark 4.12. To prove formula (4.16) one would start with the Kummer's equation (A.11) for $M(\frac{\rho}{2b}, \frac{1}{2}, z)$ and by the change of variables $z = \frac{b}{\sigma^2}x^2$ reduce this equation to the form (4.15).

We see that for $x > 0$ the function $\varphi_\rho^-(x)$ is just $U(\frac{\rho}{2b}, \frac{1}{2}, \frac{b}{\sigma^2}x^2)$, where U is the second solution to the Kummer's differential equation (A.13). As we see in the next section, functions M and U are related to the CIR process , since the square of Ornstein-Uhlenbeck $Y_t = X_t^2$ is a particular case of CIR process:

$$dY_t = (\sigma^2 - 2bY_t)dt + 2\sigma\sqrt{Y_t}dW_t.$$

The asymptotics of $\varphi_\rho^+(x)$ as $x \to \infty$ can be found using formula (A.15):

$$\varphi_\rho^+(x) \sim Cx^{\frac{\rho}{b}-1}e^{\frac{b}{\sigma^2}x^2}, \text{ as } x \to \infty; \quad \varphi_\rho^+(x) \sim Cx^{-\frac{\rho}{b}}, \text{ as } x \to -\infty \tag{4.17}$$

The asymptotics of $\varphi_\rho^-(x) = \varphi_\rho^+(-x)$ is obvious.

The boundary behavior of the process X^h and Y is given by the following lemma:

Lemma 4.13. *Let $h(x) = c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)$. Then both boundaries are natural for the process X_t^h (the same for the process Y_t).*

Proof. Let's prove the result for the right boundary $D^2 = \infty$. The result for $D^1 = -\infty$ follows by symmetry.

Let $c_1 > 0$. From lemma 3.8 and equations (B.2) we find that the speed measure and the scale function of X^h have the following asymptotics as $x \to \infty$:

$$m_{X^h}(x) = h^2(x)m_X(x) \sim Cx^{2q}e^{\frac{b}{\sigma^2}x^2}, \quad s'_{X^h}(x) = h^{-2}(x)s'_X(x) \sim Cx^{-2q}e^{-\frac{b}{\sigma^2}x^2},$$

where $q = \frac{\rho}{b} - 1$. Now using lemma 2.7 one can check that $D^2 = \infty$ is a natural boundary. The case $c_1 = 0$ and $c_2 > 0$ can be analyzed similarly. □

Thus we see that the processes Y_t associated with the Ornstein-Uhlenbeck process behave similar to processes associated with Brownian motion (quadratic volatility family): these are conservative processes. The next two examples of processes illustrate different boundary behavior: the family associated with the CIR process has one killing (exit) boundary, while the family associated with the Jacobi process has both killing (exit) boundaries.

4.4. **CIR processes - confluent hypergeometric family of driftless processes.** Let X_t be the CIR process

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dWt,$$

considered in section B.2. We will use the notations from section B.2:

$$\alpha = \frac{2a}{\sigma^2} - 1 \quad \text{and} \quad \theta = \frac{2b}{\sigma^2}.$$

Functions φ_ρ^+ and φ_ρ^- for the CIR process are solution to the ODE

$$\frac{1}{2}\sigma^2 x\frac{d^2\varphi(x)}{dx^2} - (a - bx)\frac{d\varphi(x)}{dx} = \rho\varphi(x). \tag{4.18}$$

Making affine change of variables $y = \theta x$ and dividing this equation by b, we reduce equation (4.18) to the Kummer differential equation (A.11), thus φ_ρ^+ and φ_ρ^- satisfy

$$\varphi_\rho^+(x) = M(\frac{\rho}{b}, \alpha + 1, \theta x), \tag{4.19}$$

$$\varphi_\rho^-(x) = U(\frac{\rho}{b}, \alpha + 1, \theta x). \tag{4.20}$$

Using formulas (A.13) and (A.15) we find the asymptotics of $\varphi_\rho^+(x)$ and $\varphi_\rho^-(x)$:

$$\varphi_\rho^+(x) \sim 1, \text{ as } x \to 0; \quad \varphi_\rho^+(x) \sim Ce^{\theta x}x^{\frac{\rho}{b}-\alpha-1}, \text{ as } x \to \infty; \tag{4.21}$$

$$\varphi_\rho^-(x) \sim Cx^{-\alpha}, \text{ as } x \to 0; \quad \varphi_\rho^-(x) \sim Cx^{-\frac{\rho}{b}}, \text{ as } x \to \infty. \tag{4.22}$$

The following lemma describes the boundary behavior of the process X^h (and thus of the process $Y_t = Y(X^h)$).

Lemma 4.14. *Let $h(x) = c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)$, where both c_i are positive. Then $D^2 = \infty$ is a natural boundary of the process X^h, while $D^1 = 0$ is a killing boundary if $\alpha \in (0, 1)$ and it is an exit boundary if $\alpha \geq 1$.*

Proof. Using formulas (4.21),(B.9) and lemma 3.8, we find that the asymptotics of the speed measure and scale function of X^h are

$$m_{X^h}(x) = h^2(x) m_X(x) \sim \begin{cases} Cx^{\frac{2\rho}{b}-\alpha-2}e^{\theta x}, & \text{as } x \to \infty \\ Cx^{-\alpha}, & \text{as } x \to 0, \end{cases}$$

and

$$s'_{X^h}(x) = h^{-2}(x) s'_X(x) \sim \begin{cases} Cx^{-\frac{2\rho}{b}+\alpha-1}e^{-\theta x}, & \text{as } x \to \infty \\ Cx^{\alpha-1}, & \text{as } x \to 0. \end{cases}$$

One can check using Feller's theorem 2.7, that for $\alpha \in (0,1)$ we have a regular boundary, but since the h-transform introduces nonzero killing measure at the boundaries given by equation (3.12), we have a killing boundary. If $\alpha \geq 1$ we have an exit boundary. □

Note that since the left boundary $D^1 = 0$ is either a killing or an exit boundary, the process X^h is not a conservative process. The same is true for $Y_t = Y(X^h_t)$.

Theorem 4.15. *The family of driftless processes Y_t related to a CIR process by a stochastic transformation is characterized by their volatility functions as follows:*

$$\sigma_Y(Y(x)) = C\sqrt{x}\frac{x^{-\alpha-1}e^{\theta x}}{(c_1\varphi^+_\rho(x) + c_2\varphi^-_\rho(x))^2}, \tag{4.23}$$

$$Y(x) = \frac{c_3\varphi^+_\rho(x) + c_4\varphi^-_\rho(x)}{c_1\varphi^+_\rho(x) + c_2\varphi^-_\rho(x)}.$$

Definition 4.16. We will call this family of driftless processes *the confluent hypergeometric family* and denote it by

$$\mathfrak{M}(\text{CIR}) = \{Y_t : Y \sim \text{CIR process}\}.$$

4.5. **Jacobi processes - hypergeometric family of driftless processes.** Let X_t be the Jacobi process

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t(A - X_t)}dW_t,$$

described in section B.3. Recall the notations from section B.3. We defined parameters

$$\alpha = \frac{2b}{\sigma^2} - \frac{2a}{\sigma^2 A} - 1 \text{ and } \beta = \frac{2a}{\sigma^2 A} - 1.$$

We assume that $\alpha > 0$ and $\beta > 0$, thus both boundaries are inaccessible (see section B.3).

Functions φ^+_ρ and φ^-_ρ for the Jacobi process are solutions to the ODE

$$\frac{1}{2}\sigma^2 x(A-x)\frac{d^2\varphi(x)}{dx^2} - (a - bx)\frac{d\varphi(x)}{dx} = \rho\varphi(x).$$

By the affine change of variables $x = Ay$ this equation is reduced to the hypergeometric differential equation (A.2), thus using equations (A.8) we find that functions φ^+_ρ and φ^-_ρ for the Jacobi process are given by

$$\varphi^+_\rho(x) = {}_2F_1(\alpha_1, \alpha_2; \beta_1; x/A) \tag{4.24}$$

$$\varphi^-_\rho(x) = {}_2F_1(\alpha_1, \alpha_2; \alpha_1 + \alpha_2 + 1 - \beta_1; 1 - x/A) \tag{4.25}$$

where the parameters satisfy the following system of equations:

$$\begin{cases} \alpha_1 + \alpha_2 + 1 = \frac{2b}{\sigma^2} \\ \alpha_1\alpha_2 = \frac{2\rho}{\sigma^2} \\ \beta_1 = \frac{2a}{A\sigma^2}. \end{cases} \tag{4.26}$$

The asymptotics of $\varphi^+_\rho(x)$ and $\varphi^-_\rho(x)$ are

$$\varphi^+_\rho(x) \sim 1, \text{ as } x \to 0, \quad \varphi^+_\rho(x) \sim C(A-x)^{-\alpha}, \text{ as } x \to A, \tag{4.27}$$

$$\varphi^-_\rho(x) \sim Cx^{-\beta}, \text{ as } x \to 0, \quad \varphi^-_\rho(x) \sim 1, \text{ as } x \to A. \tag{4.28}$$

Let $h(x) = c_1\varphi_\rho^+(x) + c_2\varphi_\rho^-(x)$, where both c_i are positive. The boundary behavior of the h-transformed process X_t^h is similar to the case of CIR process at $x=0$:

Lemma 4.17. *$D^1 = 0$ is a killing boundary for the process X_t^h if $\beta \in (0,1)$ and it is an exit boundary if $\beta \geq 1$. The same is true for $D^2 = A$ by changing $\beta \mapsto \alpha$.*

We see that in the case of the Jacobi process both boundaries are either killing or exit boundaries for X_t^h, thus X_t^h and Y_t are not conservative processes.

Theorem 4.18. *The family of driftless processes Y_t related to a Jacobi process by a stochastic transformation is characterized by their volatility functions as follows:*

$$\sigma_Y(Y(x)) = C\sqrt{x(A-x)}\frac{x^{-\alpha-1}(A-x)^{-\beta-1}}{(c_1\varphi_\rho^+(x)+c_2\varphi_\rho^-(x))^2}, \tag{4.29}$$
$$Y(x) = \frac{c_3\varphi_\rho^+(x)+c_4\varphi_\rho^-(x)}{c_1\varphi_\rho^+(x)+c_2\varphi_\rho^-(x)}.$$

Definition 4.19. We call this family of driftless processes the *the hypergeometric family* and denote it by

$$\mathfrak{M}(\text{Jacobi}) = \{Y_t : Y \sim \text{Jacobi process}\}.$$

5. Classification of driftless diffusion processes

In the introduction to section 3 we discuss the transformations of diffusion processes that preserve the solvability property. As we saw these transformations consist of change of variables and Doob's h-transform, which also can be considered as a change of variables and gauge transformation of the Markov generator. In this section we will focus on Markov generators. Let's ask a question: how can one transform operator $\mathcal{L}$ and solutions to the eigenfunction equation?

Let $\mathcal{L}$ be the second order differential operator

$$\mathcal{L} = a(y)\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}. \tag{5.1}$$

If $a(y)$ is positive on some interval D operator $\mathcal{L}$ can be considered as a generator of a diffusion process, thus we will call $a(y)$ the *volatility coefficient* and $b(y)$ the *drift coefficient* of operator $\mathcal{L}$.

Let's consider the following three types of transformations of the operator $\mathcal{L}$:

(i) Change of variables $x = x(y)$: the solution of the eigenfunction equation $\mathcal{L}f = \rho f$ is mapped into $f(y) \mapsto f(y(x))$ and

$$\mathcal{L} \mapsto T_{y\to x}\mathcal{L} = a(y)(x'(y))^2\frac{\partial^2}{\partial x^2} + (a(y)x''(y) + b(y)x'(y))\frac{\partial}{\partial x} = \tag{5.2}$$
$$a(y)(x'(y))^2\frac{\partial^2}{\partial x^2} + (\mathcal{L}x)(y)\frac{\partial}{\partial x}, \tag{5.3}$$

where $y = y(x)$.

(ii) Gauge transformation: $f(y) \mapsto f(y)/h(y)$ and

$$\mathcal{L} \mapsto T_h\mathcal{L} = \tfrac{1}{h}\mathcal{L}h = a(y)\frac{\partial^2}{\partial y^2} + \left(b(y) + 2a(y)\frac{h'(y)}{h(y)}\right)\frac{\partial}{\partial y} + \frac{1}{h}\left(a(y)h''(y) + b(y)h'(y)\right) =$$
$$= a(y)\frac{\partial^2}{\partial y^2} + \left(b(y) + 2a(y)\frac{h'(y)}{h(y)}\right)\frac{\partial}{\partial y} + \frac{1}{h(y)}(\mathcal{L}h)(y). \tag{5.4}$$

Notice that gauge transformation actually consists of two transformations: right multiplication of $\mathcal{L}$ by h and left multiplication by $1/h$.

(iii) Left multiplication by a function $\gamma^2(x)$ (which does not affect $f(x)$):

$$\mathcal{L} \mapsto T_{\gamma^2}\mathcal{L} = \gamma^2(y)a(y)\frac{\partial^2}{\partial y^2} + \gamma^2(y)b(y)\frac{\partial}{\partial y}. \tag{5.5}$$

In the case when $a(y) = \frac{1}{2}\sigma^2(y)$ and $\mathcal{L}$ is a generator of a Markov diffusion (Y_t, P), we can give a probabilistic interpretation to the first and second types of transformations described above: $T_{y\to x}$ is just the usual change of variables formula for the stochastic process Y_t, which describes the dynamics of the process $X_t = X(Y_t)$, thus it is just an analog of the Ito formula written in the language of ODEs. The Gauge transformation has a probabilistic meaning if $h(Y_t)$ can be considered as a measure change density (thus $h(Y_t)$ is a local martingale and $\mathcal{L}h = 0$), or when h is a ρ-excessive function ($\mathcal{L}h = \rho h$) – then the gauge transformation $\frac{1}{h}\mathcal{L}h - \rho$ is the Doob's h-transform discussed in section 3.2. The transformed generator $\mathcal{L}$ in this case describes the dynamics of the process Y_t under the new measure Q, defined by $d\mathrm{Q}_t = h(Y_t)d\mathrm{P}_t$. Note that both of these transformations preserve the form of backward Kolmogorov equation:

$$\frac{\partial}{\partial t} f(t, y) = \mathcal{L} f(t, y).$$

The last transformation T_{γ^2} does not preserve the form of backward Kolmogorov equation, thus in general it has no immediate probabilistic meaning, except when $\gamma^2(x) = c$ is constant – then T_{γ^2} is equivalent to scaling of time $t \mapsto \frac{1}{c}t'$.

As we have seen in section 3, the OU,CIR and Jacobi families of processes are solvable because they can be reduced to some simple solvable process. In other words, for these processes, the eigenfunction equation

$$\mathcal{L}_Y f(y) = \frac{1}{2}\sigma_Y^2(y)\frac{\partial^2 f(y)}{\partial y^2} = \rho f(y) \tag{5.6}$$

can be reduced by a gauge transformation (change of measure) and a change of variables to a hypergeometric or a confluent hypergeometric equation, thus giving us eigenfunctions $\psi_n(x)$, generalized eigenfunctions φ_λ^+, φ_λ^-, and a possibility to compute the transitional probability density as

$$p_Y(t, y_0, y_1) = \frac{e^{-\rho t}}{h(x_0)h(x_1)} p_X(t, x_0, x_1) = \frac{1}{h(x_0)h(x_1)} \sum_{n=0}^{\infty} e^{-(\rho-\lambda_n)t}\psi_n(x_0)\psi_n(x_1). \tag{5.7}$$

It is known that OU, CIR and Jacobi processes are the only diffusions associated with a system of orthogonal polynomials (see [11]), thus corresponding families of driftless processes are the only ones that can have a probability density of the form (5.7) where the orthogonal basis $\{\psi_n\}_{n\ge 0}$ is given by orthogonal polynomials. However we might hope to find new families of solvable processes if we generalize the definition of solvability.

Note that the fact that we can compute solutions of equation (5.6) gives us a ready expression for the Green function through the formula (2.19):

$$G_Y(\lambda, y_0, y_1) = \begin{cases} w_\lambda^{-1}\varphi_\lambda^+(y_0)\varphi_\lambda^-(y_1), & y_0 \le y_1 \\ w_\lambda^{-1}\varphi_\lambda^+(y_1)\varphi_\lambda^-(y_0), & y_1 \le y_0. \end{cases} \tag{5.8}$$

Since the Green function is a Laplace transform of $p_Y(t, y_0, y_1)$ one could hope to find the probability kernel through the inverse Laplace transform of $G_Y(\lambda, y_0, y_1)$. Thus in this section we will use the following definition of solvability:

Definition 5.1. The one dimensional diffusion process Y_t on the interval D_y is called *solvable*, if its Green function can be computed in terms of (scaled confluent) hypergeometric functions.

In other words, the process Y_t is solvable if there exist a λ-independent change of variables $y = y(z)$ and a (possibly λ-dependent) function $h(z, \lambda)$, such that all solutions to the equation

$$\mathcal{L}_Y f(y) = \lambda f(y) \tag{5.9}$$

are of the form $h(z(y), \lambda)F(z(y))$, where F is either a hypergeometric function ${}_2F_1(a, b; c; z)$, or a scaled confluent hypergeometric function $M(a, b, wz)$ (with parameters depending on λ).

Remark 5.2. Later we will see that the requirement that the change of variables is independent of λ is necessary, because otherwise every diffusion process is solvable (see remark (5.14)).

5.1. **Liouville transformations and Bose invariants.** Consider the linear second order differential operator

$$\mathcal{L}_y = a(y)\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}. \tag{5.10}$$

Making the gauge transformation with gauge factor h

$$h(y) = \exp\left(-\int^y \frac{b(u)}{2a(u)}du\right) = \sqrt{W(y)},$$

we can remove the "drift" coefficient and thereby arrive at the operator

$$\mathcal{L}_y \mapsto \frac{1}{h}\mathcal{L}_y h = a(y)\frac{\partial^2}{\partial y^2} + a(y)I(y).$$

Multiplying this operator by $a(y)^{-1}$ we arrive at symmetric operator

$$\frac{1}{h}\mathcal{L}_y h \mapsto a^{-1}\frac{1}{h}\mathcal{L}_y h = \mathcal{L}^c = \frac{\partial^2}{\partial y^2} + I(y), \tag{5.11}$$

where the potential term is given by:

$$I(y) = \left(\frac{h'(y)}{h(y)}\right)' - \left(\frac{h'(y)}{h(y)}\right)^2 = \frac{2b(y)a(y)' - 2a(y)b(y)' - b(y)^2}{4a(y)^2}. \tag{5.12}$$

Definition 5.3. $\mathcal{L}^c$ is called the *canonical form* of the operator $\mathcal{L}_y$.

The form of operator $\mathcal{L}^c$ is clearly invariant with respect to any of the three types of transformations described above. Moreover, the following lemma is true:

Lemma 5.4. *The canonical form of the operator $\mathcal{L}^c$ given by (5.11) is invariant under any two transformations of $\{T_{y\to z}, T_h, T_{\gamma^2}\}$.*

Proof. This lemma is proved by checking that every combination of two transformations cannot change the canonical form of the operator (5.11). For example, suppose we are free to use $T_{y\to z}$ and T_h, but not T_{γ^2}. Using formula (5.4) we find that by applying T_h we have nonzero drift given by $2h'/h$. We can't remove this drift by some change of variables $T_{y\to x}$, since by formula (5.2) it will add a nontrivial volatility term $x'(y)^2$, which in turn can not be removed by any gauge transformation T_h. As another example let's assume that we can use $T_{y\to x}$ and T_{γ^2}, but not T_h. By formula (5.5) we see that T_{γ^2} adds nontrivial volatility, which can be removed by $T_{y\to x}$, but formula (5.2) tells us that this in turn will add nonzero drift $\gamma^2(x)(\mathcal{L}x)(y)$, which can't be removed by any T_{γ^2}. The last combination T_{γ^2} and T_h can be checked in exactly the same way. □

Remark 5.5. Note that to bring $\mathcal{L}_y$ to canonical form $\mathcal{L}^c$ we used two types of transformations: a gauge transformation and a change of variables. But by using other choices of two transformations we could bring $\mathcal{L}_y$ to a different canonical form. Thus when we talk about canonical form we need to specify with respect to which two types of transformations this form is invariant (in this section we use only two types of canonical forms: one described above, and the second obtained by a gauge transformation and a change of variables only).

Definition 5.6. Function $I(y)$ is called the *Bose invariant* of operator (5.10) (with respect to transformations T_h and T_{γ^2}).

As we proved, function $I(y)$ is invariant with respect to any two transformations of $\{T_{y\to x}, T_h, T_{\gamma^2}\}$. However it is possible to change the potential $I(y)$ by applying all three types of transformations. The idea is to apply first a change of variables transformation, then remove the "drift" by a gauge transformation, after which we divide by the volatility coefficient to obtain a new canonical form. The details are:

(i) A change of the independent variable $y = y(x)$ (see equation (5.2)) changes the operator (5.11) into

$$\mathcal{L}_x = (y'(x))^{-2}\frac{\partial^2}{\partial x^2} - \frac{y''(x)}{(y'(x))^3}\frac{\partial}{\partial x} + I(y(x)).$$

(ii) Multiplying the operator $\mathcal{L}_x$ by $\gamma^2(x) = (y'(x))^2$ we arrive at

$$(y'(x))^2\mathcal{L}_x = \frac{\partial^2}{\partial x^2} - \frac{y''(x)}{y'(x)}\frac{\partial}{\partial x} + (y'(x))^2 I(y(x)),$$

(iii) Applying the gauge transformation with gauge factor $h = \sqrt{y'(x)}$ (see equation (5.4)) brings us to the operator in the following canonical form

$$\mathcal{L}^c_x = \frac{\partial^2}{\partial x^2} + J(x),$$

where the potential term is transformed as

$$J(x) = \frac{1}{2}\{y, x\} + (y'(x))^2 I(y(x)), \tag{5.13}$$

and $\{y, x\}$ is the *Schwarzian derivative of y with respect to x*:

$$\{y, x\} = \left(\frac{y''(x)}{y'(x)}\right)' - \frac{1}{2}\left(\frac{y''(x)}{y'(x)}\right)^2.$$

As we have seen the above transformation changes the canonical form of the operator by applying all three types of transformations. It is called a *Liouville transformation*.

Note that the order of the different steps in Liouville transformation does not matter, since all the three transformations $\{T_{y\to x}, T_h, T_{\gamma^2}\}$ commute. The other important idea is that we are free to choose the first transformation, but the other two are uniquely defined by the first one (see lemma (5.4)). We will use this fact in the proof of the main theorem in the next section.

As the first application of the canonical forms and Bose invariants we will prove the following lemma, which gives a convenient criterion to check whether two processes are related by a stochastic transformation:

Lemma 5.7. *Let X_t be a diffusion process with Markov generator*

$$\mathcal{L}_X f(x) = b(x)\frac{df(x)}{dx} + \frac{1}{2}\sigma^2(x)\frac{d^2 f(x)}{dx^2}.$$

With this diffusion we will associate a function $J_X(z) = I_X(x(z))$, where

$$I_X(x) = \frac{1}{4}\left(\sigma(x)\sigma''(x) - \frac{1}{2}(\sigma'(x))^2 + 2\left[\frac{2b(x)\sigma'(x)}{\sigma(x)} - b'(x) - \frac{b^2(x)}{\sigma^2(x)}\right]\right). \tag{5.14}$$

and the change of variables $x(z)$ is defined through $x'(z) = \sigma(x)$. Then the function $I_X(z)$ is an invariant of the diffusion X_t in the following sense:

(i) $Y_t = (Y(X_t), \mathrm{P})$ *if and only if $J_Y(z) = J_X(z)$ for all z.*
(ii) $X^h_t = (X_t, \mathrm{P}^h)$ *is an ρ-excessive transform of X_t if and only if $J_{X^h}(z) = J_X(z) - \rho$.*
(iii) $X \sim Y$ *if and only if $J_Y(z) = J_X(z) - \rho$, where ρ is the same as in stochastic transformation $\{\rho, h, Y\}$ which relates X and Y.*

Proof. Applying the gauge transformation with gauge factor $h_1(x) = \sqrt{W(x)}$ changes $\mathcal{L}_X$ as follows:

$$\mathcal{L}_X \mapsto \frac{1}{h_1}\mathcal{L}_X h_1 = \frac{1}{2}\sigma^2_X(x)\frac{\partial^2}{\partial x^2} + I_1(x), \tag{5.15}$$

where by formula (5.12) the potential term is equal

$$I_1(x) = \frac{1}{2}\left[\frac{2b_X(x)\sigma'_X(x)}{\sigma_X(x)} - b'_X(x) - \frac{b^2_X(x)}{\sigma^2_X(x)}\right]. \tag{5.16}$$

Changing variables $x'(z) = \frac{1}{\sqrt{2}}\sigma_X(x)$ we arrive at

$$\mathcal{L}_z = \frac{\partial^2}{\partial z^2} - \frac{x''(z)}{x'(z)}\frac{\partial}{\partial z} + I_1(x(z)), \tag{5.17}$$

and at last making the gauge transformation with gauge factor $h_2(z) = \sqrt{x'(z)}$ we arrive at the canonical form

$$\mathcal{L}^c = \frac{1}{h_2}\mathcal{L}_z h_2 = \frac{\partial^2}{\partial z^2} + J_X(z) \tag{5.18}$$

where the Bose invariant (potential term) is

$$J_X(z) = \frac{1}{2}\{x, z\} + I_1(x). \tag{5.19}$$

By direct computations we check that $\{x, z\} = \frac{1}{2}\sigma''_X(x)\sigma_X(x) - \frac{1}{4}\sigma^2_X(x)$, which gives us formula (5.14).

Now we need to prove that $J_X(z)$ is invariant under stochastic transformations up to an additive constant. A change of variables does not change the form of $J_X(z)$. A change of measure (ρ-excessive transform) changes the generator of X

$$\mathcal{L}_X \mapsto \frac{1}{h}\mathcal{L}_X h - \rho.$$

Since $J_X(z)$ is invariant under gauge transformations $\mathcal{L}_X \mapsto \frac{1}{h}\mathcal{L}_X h$ we see that $J_Y(z)$ differs from $J_X(z)$ by a constant $-\rho$, which ends the proof. □

Example 5.8. As we showed in the previous section the *quadratic volatility family* with volatility functions of the form

$$\sigma_Y(y) = a_2 y^2 + a_1 y + a_0 \tag{5.20}$$

is related to a Brownian motion $X_t = W_t$ by a stochastic transformation. It is very easy to prove this fact using the above lemma:

$$J_W \equiv 0, \qquad\qquad J_Y \equiv \text{const},$$

thus $Y_t \sim W_t$.

Example 5.9. As another application of lemma 5.7 let's show that a Bessel process

$$dX_t = a dt + \sqrt{X_t} dW_t$$

is related to CEV processes (constant-elasticity-of-variance)

$$dY_t = Y_t^{\theta} dW_t.$$

Note that in subsection 4.2 we constructed these stochastic transformations explicitly.

Function $I_X(x)$ is given by

$$I_X(x) = \frac{1}{4}\left(-\frac{1}{4x} - \frac{1}{8x} + 2\left[\frac{a}{x} - \frac{a^2}{x}\right]\right) = \frac{1}{4x}(-2a^2 + 2a - 3/8)$$

After making the change of variables $dx = \sigma_X(x)dz = \sqrt{x}dz$ (thus $x = z^2/4$) we arrive at

$$J_X(z) = z^{-2}(-2a^2 + 2a - 3/8).$$

Similarly for the process Y_t

$$dy = \sigma_Y(y)dz = y^{\theta}dz \Rightarrow y = ((1-\theta)z)^{\frac{1}{1-\theta}}$$

and

$$J_Y(z) = \frac{1}{4}\left(\theta(\theta-1)y^{2\theta-2} - \frac{1}{2}\theta^2 y^{2\theta-2}\right) = \frac{1}{4}z^{-2}\frac{\frac{\theta^2}{2} - \theta}{(1-\theta)^2}.$$

One can check that when $\theta = 1 - \frac{1}{2(2a-1)}$ we have $J_Y(z) = J_X(z)$, thus processes Y_t should be related to the Bessel process by a stochastic transformation $\{\rho, h, Y\}$ with $\rho = 0$ (see section 4.2 where we construct this transformation explicitly).

The following diagram illustrates the importance of a Liouville transformation (in particular the T_{γ^2} transformation). Assume that we start with a diffusion process X_1. By stochastic transformations we can construct the family $\mathfrak{M}(X_1)$ of driftless processes. By a gauge transformation and by a change of variables we map the generator of X_1 into the corresponding canonical form (with the Bose invariant given by (5.14)). By applying a Liouville transformation we change the potential (Bose invariant) of the canonical form and then by a change of variables and a gauge transformation we can map it back into the generator of some diffusion process X_2. Then as before we construct a family of driftless processes $\mathfrak{M}(X_2)$. Note that $\mathfrak{M}(X_2)$ and $\mathfrak{M}(X_1)$ are not related by a stochastic canonical transformation, since otherwise they would have the same Bose invariants. Then the process can be continued. Thus we have a family of Bose invariants related by Liouville transformations, each of these invariants (potentials in canonical form) gives rise to a new family of driftless processes, which are not related by a stochastic transformation to the previous families (otherwise their Bose invariants would coincide).

The following diagram summarizes the above ideas:

$\mathfrak{M}(X_1)$		$\mathfrak{M}(X_2)$		$\mathfrak{M}(X_3)$
$\wr$		$\wr$		$\wr$
stochastic transformations		stochastic transformations		stochastic transformations
$\wr$		$\wr$		$\wr$
X_1	$\not\sim$	X_2	$\not\sim$	X_3
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$J_{X_1}(z)$	$\Longleftrightarrow$	$J_{X_2}(z)$	$\Longleftrightarrow$	$J_{X_3}(z)$
	Liouville transformation		Liouville transformation	

5.2. **Classification: Main theorems.**

Theorem 5.10. First classification theorem. *A driftless process Y_t is solvable in the sense of definition 5.1 if and only if its volatility function is of the following form:*

$$\sigma_Y(y) = \sigma_Y(Y(x)) = C\sqrt{A(x)}\frac{W(x)}{(c_1F_1(x)+c_2F_2(x))^2}\sqrt{\frac{A(x)}{R(x)}}, \tag{5.21}$$

where the change of variables is given by

$$y = Y(x) = \frac{c_3F_1(x)+c_4F_2(x)}{c_1F_1(x)+c_2F_2(x)}, \quad c_1c_4 - c_3c_2 \neq 0. \tag{5.22}$$

In the case $A(x) = x$:

(i) *$R(x) \in P_2$, such that $R(x) \neq 0$ in $(0,\infty)$*
(ii) *F_1 and F_2 are functions $M(a,b,wx)$ and $U(a,b,wx)$*
(iii) *$W(x)$ is a Wronskian of the scaled Kummer differential equation (equal to $s'(x)$ in (B.9)).*

and in the case $A(x) = x(1-x)$

(i) *$R(x) \in P_2$, such that $R(x) \neq 0$ in $(0,1)$*
(ii) *F_1 and F_2 are two linearly independent solutions to the hypergeometric equation given by (A.3).*
(iii) *$W(x)$ is a Wronskian of the hypergeometric differential equation (equal to $s'(x)$ in (B.17)).*

Before we prove this theorem we need to establish some auxiliary results.

Lemma 5.11. *The Bose invariant for the hypergeometric equation*

$$x(1-x)\frac{\partial^2 f(x)}{\partial x^2} + (\gamma - (1+\alpha+\beta)x)\frac{\partial f(x)}{\partial x} - \alpha\beta f(x) = 0.$$

is given by

$$I_{hyp}(x) = \frac{Q(x)}{4x^2(1-x)^2}, \tag{5.23}$$

where

$$Q(x) = (1-(\alpha-\beta)^2)x^2 + (2\gamma(\alpha+\beta-1) - 4\alpha\beta)x + \gamma(2-\gamma). \tag{5.24}$$

The Bose invariant for the scaled confluent hypergeometric equation

$$x\frac{\partial^2 f(x)}{\partial x^2} + (a - wx)\frac{\partial f(x)}{\partial x} - bwf(x) = 0$$

is given by

$$I_{confl}(x) = \frac{Q(x)}{4x^2}, \tag{5.25}$$

where

$$Q(x) = -w^2x^2 + 2w(a-2b)x + a(2-a). \tag{5.26}$$

In both cases by varying parameters we can obtain any second order polynomial Q.

Proof. For the second order differential equation

$$a(x)\frac{\partial^2 f(x)}{\partial x^2} + b(x)\frac{\partial f(x)}{\partial x} + c(x)f(x) = 0 \tag{5.27}$$

the Bose invariant is given by formula (5.12) and is equal to

$$\begin{aligned} I(x) &= \frac{2b(x)a(x)' - 2a(x)b(x)' - b(x)^2}{4a(x)^2} + \frac{c(x)}{a(x)} \\ &= \frac{2b(x)a(x)' - 2a(x)b(x)' - b(x)^2 + 4a(x)c(x)}{4a(x)^2}. \end{aligned}$$

□

Corollary 5.12. *Let $T(x) \in P_2$ be an arbitrary second order polynomial and $A(x) \in \{x, x(1-x)\}$. The solutions to equation*

$$\frac{\partial^2 f(x)}{\partial x^2} + \frac{T(x)}{A^2(x)}f(x) = 0$$

can be obtained in the form $F(x)/\sqrt{W(x)}$, where $W(x)$ is the Wronskian given by $\exp(-\int^x \frac{b(y)}{a(y)}dy)$ and function $F(x)$ is a solution to the hypergeometric equation in the case $A(x) = x(1-x)$ or to the scaled confluent hypergeometric equation in the case $A(x) = x$.

Theorem 5.13. (Schwarz) *The general solution to the equation*

$$\frac{1}{2}\{y, x\} = J(x) \tag{5.28}$$

has the form

$$y(x) = \frac{F_2(x)}{F_1(x)}, \tag{5.29}$$

where F_1 and F_2 are arbitrary linearly independent solutions of equation

$$\frac{\partial^2 F(x)}{\partial x^2} + J(x)F(x) = 0. \tag{5.30}$$

Proof. Let's introduce the new variable $F(x) = \frac{1}{\sqrt{y'(x)}}$. Then we have

$$y'(x) = \frac{1}{(F(x))^2} \quad \Rightarrow \log y'(x) = -2\log(F(x)) \quad \Rightarrow \frac{y''(x)}{y'(x)} = -2\frac{F'(x)}{F(x)}.$$

Thus the Schwarzian derivative $\{y, z\}$ is equal to:

$$\{y, z\} = \left(\frac{y''(x)}{y'(x)}\right)' - \frac{1}{2}\left(\frac{y''(x)}{y'(x)}\right)^2 = -2\frac{F''(x)}{F(x)}$$

and equation (5.28) is transformed into equation

$$F''(x) + J(x)F(x) = 0$$

for function $F(x)$, thus

$$y'(x) = \frac{1}{(F_1(x))^2},$$

where $F_1(x)$ is an arbitrary solution of $F''(x) + J(x)F(x) = 0$.

Note that if F_1 and F_2 are solutions to second order linear ODE $aF'' + bF' + cF = 0$, then

$$\frac{d}{dx}\left(\frac{F_2(x)}{F_1(x)}\right) = \frac{F_2'(x)F_1(x) - F_2(x)F_1'(x)}{F_2^2(x)} = \frac{W_{F_2,F_1}(x)}{F_1^2(x)},$$

and since the Wronskian of equation (5.30) is constant we can obtain the above expression for $y(x)$. □

Remark 5.14. Note that theorem 5.13 and equation (5.13) tells us that for any potential there exists a change of variables $z(x)$, which maps equation $F'' = 0$ into equation $F''(x) + J(x)F(x) = 0$, thus any two canonical forms can be related by some Liouville transformation. That is why in the definition (5.1) we require the change of variables $y(x)$ to be independent of λ.

Proof of the first classification theorem:

Proof. By definition 5.1 the process Y_t is solvable if for all λ we can reduce equation

$$\mathcal{L}_Y f(y) = \frac{1}{2}\sigma_Y^2(y)\frac{\partial^2 f(y)}{\partial y^2} = \lambda f(y)$$

to the (scaled confluent) hypergeometric equation by a Liouville transformation, with a change of variables $y = y(x)$ independent of λ. The Liouville transformation consists of three parts: a change of variables, a multiplication by function and a gauge transformation and all these transformations commute. For example we could first apply a multiplication by function transformation, and then use only change of variables and gauge transformations. Thus we can reformulate the problem as follows: find all functions $\sigma_Y(y)$, such that there exist a function $\gamma(y)$, such that the equation

$$2\gamma^2(y)\mathcal{L}_Y f(y) = \sigma_Y^2(y)\gamma^2(y)\frac{\partial^2 f(y)}{\partial y^2} = 2\lambda\gamma^2(y)f(y) \tag{5.31}$$

can be reduced to the (confluent) hypergeometric equation by a change of variables (independent of λ) and a gauge transformation. The Bose invariant of equation (5.31) is given by

$$I(x) = \frac{1}{2}\{y, x\} - 2\lambda\gamma^2(y(x)), \tag{5.32}$$

with the change of variables given by

$$\frac{dy}{dx} = \sigma_Y(y)\gamma(y). \tag{5.33}$$

Note that $\gamma(y)$ must be independent of λ since $\sigma_Y(y)$ and $y'(x)$ are independent of λ.

By lemma 5.4 we know that equation (5.31) can be reduced to the (confluent) hypergeometric equation by a change of variables and a gauge transformation if and only if the corresponding Bose invariants are equal, that is

$$I(x) = I_{hyp}(x), \quad \text{or } I(x) = I_{confl}(x),$$

thus, combining equations (5.32) with (5.23) and (5.25), we have

$$\frac{1}{2}\{y, x\} - 2\lambda\gamma^2(y(x)) = \frac{Q(x)}{A^2(x)},$$

where $A(x) \in \{x, x(1-x)\}$ and $Q(x) = Q(x, \lambda)$ is some second order polynomial in x with parameters depending on λ. Note that $\{y, x\}$ and $\gamma^2(y(x))$ are independent of λ, thus there exist two polynomials

$T(x), R(x) \in P_2$, independent of λ, such that

$$\begin{cases} Q(x) = T(x) - 2\lambda R(x), \\ \frac{1}{2}\{y,x\} = \frac{T(x)}{A^2(x)}, \\ \gamma^2(y(x)) = \frac{R(x)}{A^2(x)}, \quad R(x) \ge 0. \end{cases} \tag{5.34}$$

The last equation in the system (5.34) gives us the function $\gamma(y(x))$:

$$\gamma(y(x)) = \frac{\sqrt{R(x)}}{A(x)}. \tag{5.35}$$

By theorem 5.13, the solutions to equation $\frac{1}{2}\{y,x\} = \frac{T(x)}{A^2(x)}$ are given by

$$y(x) = \frac{c_3 f_1(x) + c_4 f_2(x)}{c_1 f_1(x) + c_2 f_2(x)}, \tag{5.36}$$

where f_1 and f_2 are linearly independent solutions to equation

$$f''(x) + \frac{T(x)}{A^2(x)} f(x) = 0. \tag{5.37}$$

Now we can use corollary (5.12), which tells us that all the solutions to equation (5.37) can be found in the form $F(x)/\sqrt{W(x)}$, thus

$$y(x) = \frac{c_3 F_1(x) + c_4 F_2(x)}{c_1 F_1(x) + c_2 F_2(x)} \tag{5.38}$$

where F_1 and F_2 are confluent hypergeometric ($A(x) = x$) or hypergeometric ($A(x) = x(1-x)$) functions.

Now we are ready to find volatility function $\sigma_Y(y)$. From the equation (5.33) we find that

$$\sigma_Y(Y(x)) = y'(x) \frac{1}{\gamma(y(x))}. \tag{5.39}$$

The derivative $y'(x)$ can be computed as

$$y'(x) = \frac{CW(x)}{(c_1 F_1(x) + c_2 F_2(x))^2},$$

thus

$$\sigma_Y(Y(x)) = y'(x) \frac{1}{\gamma(y(x))} = C \frac{CW(x)}{(c_1 F_1(x) + c_2 F_2(x))^2} \frac{A(x)}{\sqrt{R(x)}},$$

which completes the proof. □

Definition 5.15. We call the family of driftless processes with volatility function given by equation (5.21) a *hypergeometric R-family* in the case $A(x) = x(1-x)$ and a *confluent hypergeometric R-family* in the case $A(x) = x$.

We see that in the case $R(x) = A(x)$ we recover the hypergeometric and confluent hypergeometric families, which correspond to $\mathfrak{M}$(Jacobi) and $\mathfrak{M}$(CIR). In the case $R(x) \neq A(x)$ we obtain new families of processes. The next theorem shows that the (confluent) hypergeometric R-family can be obtained by stochastic transformations described from some diffusion process (in the same way as Jacobi and CIR families are generated by a single diffusion process).

Theorem 5.16. Second classification theorem. *Let $R(x) \in P_2$ be a second degree polynomial in x.*

(i) *The confluent hypergeometric case: Assume that $R(x)$ has no zeros in $(0, \infty)$. Let $X_t = X_t^R$ be the diffusion process with dynamics*

$$dX_t = (a + bX_t)\frac{X_t}{R(X_t)} dt + \frac{X_t}{\sqrt{R(X_t)}} dW_t. \tag{5.40}$$

Then the confluent hypergeometric R-family coincides with $\mathfrak{M}(X_t^R)$ and thus can be obtained from X_t^R by stochastic transformations. In the particular case $R(x) = A(x) = x$ we have X_t^R is a CIR process and we obtain the CIR family defined in (4.4).

(ii) *The hypergeometric case: Assume that $R(x)$ has no zeros in $(0, 1)$. Let $X_t = X_t^R$ be the diffusion process with dynamics:*

$$dX_t = (a + bX_t)\frac{X_t(1 - X_t)}{R(X_t)}dt + \frac{X_t(1 - X_t)}{\sqrt{R(X_t)}}dW_t. \tag{5.41}$$

Then the hypergeometric R-family coincides with $\mathfrak{M}(X_t^R)$ and thus can be obtained from X_t^R by stochastic transformations. In the particular case $R(x) = A(x) = x(1 - x)$ we have X_t^R is a Jacobi process and we obtain the Jacobi family defined in (4.5).

Proof. The generator of X_t is given by

$$\mathcal{L}_X = (a + bx)\frac{A(x)}{R(x)}\frac{\partial}{\partial x} + \frac{1}{2}\frac{A^2(x)}{R(x)}\frac{\partial^2}{\partial x^2}. \tag{5.42}$$

One way to prove this theorem is to check that the corresponding Bose invariants coincide. However we prove this theorem by applying stochastic transformations to the process X_t and showing that we can cover all of the processes with volatility functions given by equation (5.21).

First we need to find two linearly independent solutions to the "eigenfunction" equation

$$\mathcal{L}_X\varphi = \rho\varphi.$$

This equation is equivalent to

$$2(a + bx)\frac{\partial\varphi(x)}{\partial x} + A(x)\frac{\partial^2\varphi(x)}{\partial x^2} = 2\rho\frac{R(x)}{A(x)}\varphi(x).$$

By dividing both sides by $A(x)$ and making gauge transformation with gauge factor $h(x) = \sqrt{W(x)}$, $F = \varphi/h$, we arrive at the equation in canonical form:

$$\frac{\partial^2 F(x)}{\partial x^2} + \frac{Q(x) - 2\alpha R(x)}{A^2(x)}F(x) = 0.$$

By corollary 5.12 this equation is solved in terms of hypergeometric functions, thus $\varphi_i(x) = g(x)F_i(x)$, where F_i are (confluent) hypergeometric functions. Thus

$$Y(x) = \frac{c_1\varphi_1(x) + c_2\varphi_2(x)}{c_3\varphi_1(x) + c_4\varphi_2(x)} = \frac{c_1F_1(x) + c_2F_2(x)}{c_3F_1(x) + c_4F_2(x)},$$

and $\sigma_Y(y)$ is computed as

$$\sigma_Y(y) = Y'(x)\frac{A(x)}{\sqrt{R(x)}} = C\sqrt{A(x)}\frac{W(x)}{(c_1F_1(x) + c_2F_2(x))^2}\sqrt{\frac{A(x)}{R(x)}},$$

which ends the proof. □

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APPENDIX A. HYPERGEOMETRIC FUNCTIONS

In this section, we review basic notions about hypergeometric functions. A good collection of facts and formulas can be found in [8] and [1].

A.1. **Hypergeometric function.** The hypergeometric function ${}_2F_1(\alpha,\beta;\gamma;z)$ is defined through its Taylor expansion:

$$ {}_2F_1(\alpha,\beta;\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}. \tag{A.1}$$

${}_2F_1(\alpha,\beta;\gamma;z)$ is a solution to the *hypergeometric differential equation*

$$ z(1-z)F''(z)+(\gamma-(1+\alpha+\beta)z)F'(z)-\alpha\beta F(z)=0. \tag{A.2}$$

This differential equation has three (regular) singular points: $0,1,\infty$. The exponents at $z=0$ are $0,1-\gamma$ and at $z=1$ are $0,\gamma-\alpha-\beta$.

Two linearly independent solution in the neighborhood of $z=0$ are given by:

$$ w_1={}_2F_1(\alpha,\beta;\gamma;z), \tag{A.3}$$

$$ w_2=z^{1-\gamma}{}_2F_1(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z), \tag{A.4}$$

and in the neighborhood of $z=1$

$$ w_1={}_2F_1(\alpha,\beta;\alpha+\beta+1-\gamma;1-z), \tag{A.5}$$

$$ w_2=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1,1-z). \tag{A.6}$$

The derivative of the hypergeometric function is:

$$ {}_2F_1'(\alpha,\beta;\gamma;z)=\frac{\alpha\beta}{\gamma}{}_2F_1(\alpha+1,\beta+1;\gamma+1,z). \tag{A.7}$$

Increasing and decreasing solutions of the hypergeometric equation, which in the case $\alpha>0,\ \beta>0,\ \gamma>0$ and $\gamma<\alpha+\beta+1$ are given by:

$$ \varphi^+(x)={}_2F_1(\alpha,\beta;\gamma;z), \tag{A.8}$$

$$ \varphi^-(x)={}_2F_1(\alpha,\beta;\alpha+\beta+1-\gamma;1-z). \tag{A.9}$$

A.2. **Confluent hypergeometric function.** The confluent hypergeometric function ${}_1F_1(a,b,z)$ (also denoted by $M(a,b,z)$ or $\Phi(a,b,z)$) can be defined through its Taylors expansion

$$ M(a,b,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}. \tag{A.10}$$

Function $M(a,b,z)$ is a solution to the *Kummer differential equation*

$$ zF''(z)+(b-z)F'(z)-aF(z)=0 \tag{A.11}$$

It has two singular points: $0,\infty$. 0 is a regular singular point with the exponents $0,1-b$.

Two linearly independent solutions to the Kummer differential equation are given by:

$$ w_1=M(a;b;z),\quad w_2=z^{1-b}M(1+a-b;2-b;z). \tag{A.12}$$

The increasing and decreasing solutions are given by:

$$ \varphi^+(x)=M(a;b;z), \tag{A.13}$$

$$ \varphi^-(x)=U(a,b,z)=\frac{\pi}{\sin(\pi b)}\left(\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)}-z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right). \tag{A.14}$$

The asymptotics of M and U as $|z| \to \infty$ ($\Re z > 0$) is:

$$\varphi^+(z) = M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}(1 + O(|z|^{-1})), \tag{A.15}$$

$$\varphi^-(z) = U(a,b,z) = z^{-a}(1 + O(|z|^{-1})), \tag{A.16}$$

and the derivative of the confluent hypergeometric function can be computed as

$$M'(a,b,z) = \frac{a}{b} M(a+1, b+1, z). \tag{A.17}$$

APPENDIX B. ORNSTEIN-UHLENBECK,CIR, AND JACOBI PROCESSES

In this section we present some facts about the following three diffurion processes: OU, CIR and Jacobi. These processes enjoy a lot of interesting properties: first of all , these processes are associated with a family of orthogonal polynomials, which means that these polynomials form a complete set of eigenfunctions of the generator $\mathcal{L}$. Furthermore, it can be proved that these are the only diffusion processes which have this property (see [11]). This property of generator $\mathcal{L}$ allows us to express the probability semigroup $P(t)$ as an orthogonal expansion in these polynomials, thus all of these processes are solvable (moreover, there are explicit formulas for OU and CIR).

In this section we just briefly present the formulas for the generator, speed measure and scale function, describe boundary behavior, present eigenfunctions and eigenvalues of the generators and series expansion for the probability density (along with an explicit formula if possible).

B.1. **Ornstein-Uhlenbeck process.**

- Generator:

$$\mathcal{L} = (a - bx)\frac{d}{dx} + \frac{1}{2}\sigma^2 \frac{d^2}{dx^2}, \tag{B.1}$$

  where $b > 0$.
- Domain: $D = (-\infty, +\infty)$.
- Speed measure and scale function:

$$m(x) = \frac{1}{\sqrt{\pi \frac{\sigma^2}{b}}} \exp\left(-\frac{b}{\sigma^2}(x - \frac{a}{b})^2\right), \quad s'(x) = \exp\left(\frac{b}{\sigma^2}(x - \frac{a}{b})^2\right) \tag{B.2}$$

- Boundary conditions: Both $D^1 = -\infty$ and $D^2 = +\infty$ are natural boundaries for all choices of parameters.
- Probability function:

$$p^{(OU)}(t, x_0, x_1) m(x_1) = \frac{1}{\sqrt{\pi \frac{\sigma^2}{b}(1 - e^{-2bt})}} \exp\left(-\frac{\frac{b}{\sigma^2}(x_1 - x_0 e^{-bt} - \frac{a}{b}(1 - e^{-bt}))^2}{(1 - e^{-2bt})}\right). \tag{B.3}$$

- Spectrum of the generator:

$$\lambda_n = -bn. \tag{B.4}$$

- Eigenfunctions of the generator:

$$\psi_n(x) = H_n\left(\sqrt{\frac{b}{\sigma^2}}(x - \frac{a}{b})\right), \tag{B.5}$$

  where $H_n(x)$ are Hermite polynomials. The three term recurrence relation is:

$$H_{n+1} - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{B.6}$$

- Orthogonality relation

$$\int_D \psi_n(x)\psi_m(x)m(x)dx = 2^n n! \delta_{nm}.$$

- Eigenfunction expansion of the probability function:

$$p^{(\mathrm{OU})}(t,x_0,x_1)=\sum_{n=0}^{\infty}\frac{e^{-bnt}}{2^n n!}H_n(y_0)H_n(y_1), \tag{B.7}$$

where $y_i=\sqrt{\frac{b}{\sigma^2}}(x_i-\frac{a}{b})$.

B.2. **CIR process.**

- Generator

$$\mathcal{L}=(a-bx)\frac{d}{dx}+\frac{1}{2}\sigma^2 x\frac{d^2}{dx^2}. \tag{B.8}$$

- Domain $D=[0,+\infty)$
- Speed measure and scale function:

$$m(x)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}x^{\alpha}e^{-\theta x}, \quad s'(x)=x^{-\alpha-1}e^{\theta x}, \tag{B.9}$$

where $\alpha=\frac{2a}{\sigma^2}-1$ and $\theta=\frac{2b}{\sigma^2}$.

- Boundary conditions: $D^2=+\infty$ is a natural boundary for all choices of parameters and

$$D^1=\begin{cases}\text{exit, if } \alpha\le -1\\ \text{regular, if } -1<\alpha<0\\ \text{entrance, if } 0\le\alpha\end{cases} \tag{B.10}$$

- Probability function:

$$p^{(CIR)}(t,x_0,x_1)m(x_1)=c_t\left(\frac{x_1e^{bt}}{x_0}\right)^{\frac{1}{2}\alpha}\exp\left[-c_t(x_0e^{-bt}+x_1)\right]I_{\alpha}\left(2c_t\sqrt{x_0x_1e^{-bt}}\right), \tag{B.11}$$

where $c_t\equiv -2b/(\sigma^2(e^{-bt}-1))$ and I_α is the modified Bessel function of the first kind (see [8]).

- Spectrum of the generator:

$$\lambda_n=-bn. \tag{B.12}$$

- Eigenfunctions of the generator:

$$\psi_n(x)=L_n^{\alpha}(\theta x), \tag{B.13}$$

where $L_n^{\alpha}(y)$ are Laguerre polynomials of order α with the three term recurrence relation:

$$(n+1)L_{n+1}^{\alpha}(y)-(2n+\alpha+1-y)L_n^{\alpha}(y)+(n+\alpha)L_{n-1}^{\alpha}(y)=0. \tag{B.14}$$

- Orthogonality relation:

$$\int_D \psi_n(x)\psi_m(x)m(x)dx=\frac{(\alpha+1)_n}{n!}\delta_{nm}.$$

- Eigenfunction expansion of the probability function:

$$p^{(\mathrm{CIR})}(t,x_0,x_1)=\sum_{n=0}^{\infty}e^{-bnt}\frac{n!}{(\alpha+1)_n}L_n^{\alpha}(\theta x_0)L_n^{\alpha}(\theta x_1). \tag{B.15}$$

B.3. **Jacobi process.**

- Generator

$$\mathcal{L}=(a-bx)\frac{d}{dx}+\frac{1}{2}\sigma^2 x(A-x)\frac{d^2}{dx^2}. \tag{B.16}$$

- Domain $D=[0,A]$
- Speed measure and scale function:

$$m(x)=\frac{x^{\beta}(A-x)^{\alpha}}{A^{\alpha+\beta+1}B(\alpha+1,\beta+1)}, \quad s'(x)=x^{-\beta-1}(A-x)^{-\alpha-1}, \tag{B.17}$$

where $\alpha=\frac{2b}{\sigma^2}-\frac{2a}{\sigma^2 A}-1$ and $\beta=\frac{2a}{\sigma^2 A}-1$.

- Boundary behavior for the Jacobi process is the same as for CIR process at the left boundary:

$$D^1 = \begin{cases} \text{exit, if } \beta \le -1 \\ \text{regular, if } -1 < \beta < 0 \\ \text{entrance, if } 0 \le \beta \end{cases} \tag{B.18}$$

The same classification applies to right boundary, we only need to replace β by α. Notice that in the case when $a > 0$, $b > 0$ and $\frac{a}{b} < A$ (which means that mean-reverting level lies in the interval $(0, A)$), we have $\alpha > -1$ and $\beta > -1$ and thus both boundaries are not exit.

- Spectrum of the generator:

$$\lambda_n = -\frac{\sigma^2}{2} n(n - 1 + \frac{2b}{\sigma^2}). \tag{B.19}$$

- Eigenfunctions of the generator:

$$\psi_n(x) = P_n^{(\alpha,\beta)}(y), \tag{B.20}$$

where $y = (\frac{2x}{A} - 1)$ and $P_n^{(\alpha,\beta)}(y)$ are Jacobi polynomials with the three term recurrence relation:

$$\begin{aligned} yP_n^{(\alpha,\beta)}(y) = \; & \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)}(y) + \\ & + \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} P_n^{(\alpha,\beta)}(y) + \\ & + \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(y). \end{aligned}$$

- Orthogonality relation

$$\int_D \psi_n(x)\psi_m(x)m(dx) = p_n^2 \delta_{nm} = \frac{(\alpha+1)_n(\beta+1)_n}{(\alpha+\beta+2)_{n-1}(2n+\alpha+\beta+1)n!} \delta_{nm}.$$

- Eigenfunction expansion of the probability function:

$$p^{(\text{Jacobi})}(t, x_0, x_1) = \sum_{n=0}^{\infty} \frac{e^{-\lambda_n t}}{p_n^2} P_n^{(\alpha,\beta)}(y_0) P_n^{(\alpha,\beta)}(y_1), \tag{B.21}$$

where $y_i = (\frac{2x_i}{A} - 1)$.

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