Integrability by Quadratures of Pricing Equations

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Abstract

We introduce a canonical transformation method for finding solutions to pricing problems by quadratures. The method is systematic and allows one to derive in a unified framework the exact solutions in the pricing literature. As an application, we construct a new families of pricing models based on the squared Bessel process which extends the constant-variance-of-volatility (CEV) model and is integrable by quadratures.

The main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore we must travel the reverse path and after finding some notable substitution, look for problems to which it can be successfully applied.

Jacobi, "Lectures on Dynamics", 1847.

1 Introduction

Wiener processes are one of the mathematical building blocks of pricing theory. They enter either directly as in Bachelier's work [1], where the stock price process is postulated to follow a Brownian motion with drift, or indirectly as in Samuelson [18] where the exponential of a Wiener process, i.e. a geometric Brownian motion is used. The seminal works on arbitrage-free derivative pricing by Merton [12] and BlackScholes [2] is also based on geometric Brownian motion as a simple way of ensuring positivity. In fact, a put option struck at zero on a stock should always be worthless in any self-consistent model. More general models portray to capture the correlation between stock prices and volatility. In principle, state dependent volatilities can be derived non-parameterically, from the option prices themselves as in Dupire [9] and Derman-Kani [8]. Paremeterized forms of the local volatility function are however more suitable as they are more stable from the viewpoint of statistical analysis of option data. In this context, the special forms of the volatility function which give rise to analytically explicit pricing formulas are of interest as they give rise to more efficient numerical algorithms for model calibration. Among the analytically tractable volatility functions that appeared in the literature, one counts the quadratic volatility models in Bluman [4] [3], Sonderman [20], Rady and Sandman [16] [15], Ingerssol [10], Zühlsdorff [21] [22] and the constant-elasticity-of-variance (CEV) model in papers by Cox and Ross [7] and Schröder [19]. The techniques for obtaining exact solutions rely on reductions to elementary processes such as the Wiener and the Bessel squared process. By mean of Lie group methods, Bluman [4] [3] shows that quadratic volatility models represent the most general class which is reducible to the Wiener process, see also [17]. Carr, Lipton and Madan [5] obtain a new proof of this result in a setting which is more amenable to financial interpretation, by combining non-linear transformations and numeraire changes, see also Nelson and Ramaswamy [14]. In [5], it is also shown that the most general pricing equation of the Black-Scholes type for diffusion processes, reduces to martingale diffusions with time and state dependent volatility.

In this article, we consider the problem of finding conditions which ensure integrability by quadratures. We make use of combinations of measure changes and non-linear maps that we refer to as *canonical transformations* . The choice of nomenclature stems from the parallel with classical mechanics, where canonical transformations with analogous properties are used to find exactly solveable models through the Hamilton-Jacobi equations. (Notice that, although the analogy is instructive, this article is self-contained and the reader is not expected to be knowledgeable in classical mechanics.) In analogy with the theory of solveability in classical mechanics, we restrict ourselves to models and canonical transformations satisfying factorization conditions such as the following:

(i) the process for the financial observable is restricted to be a martingale with a local volatility function which factorizes into the product of a time dependent and a state dependent function;

(ii) the measure chage is given by a numeraire asset restricted by the constraint that the corresponding price process factorizes into the product of a state dependent and of a time dependent function;

(iii) the underlying follows a process with constant drift under the new numeraire.

Working with the restricted class of canonical transformations satisfying the above factorization conditions, we show that the pricing problem is reconducible to quadratures if

(a) the Fokker-Plank equation for the underlying process can be integrated by quadratures;

(b) a second order ordinary differential equation of the Shrödinger type can be solved by quadratures.

Shrödinger equations arise in quantum mechanics and are the subject of abundant literature, as well as the Fokker-Plank equations. In case boundary conditions due to barrier clauses are present, the Fokker-Plank equation can be solved by eigenfunction expansions and Laplace transform methods from the theory of heat diffusion in solids [6], as shown by Linetsky and Davidov [11] in a financial context. Our framework gives a simple tool to build pricing models which are guaranteed to be integrable by quadratures and take advantage of the exact solutions of these second order differential equations. Often, further simplifications occur as the quadratures can be explicitely carried out in terms of special functions. In this article we show that within our restrictive framework, one can recover the known integrable pricing models in the literature. As an application, we also build a new 4-parameter family of integrable models based on the square of the Bessel process and extending the CEV model in [7] and [19].

The paper is organized as follows. In Section 2 we introduce the notion of canonical transformations, derive a defining system of differential equations and reduce them to a Schrödinger equation. In Section 3, we reconduce the problem of finding a volatility function for the overlying in terms of the volatility of the underlying and the drift under the forward measure, to quadratures. In Section 4, 5 and 6 we describe three partially overlapping families of models integrable by quadratures, those reconducible to the Wiener process, those reconducible to the square of the Bessel process and those which satisfy a particular martingale condition which makes it possible to solve the Shrödinger equation by quadratures. In section 7 we derive pricing formulas for European and barrier options in terms of eigenfunction expansions and Laplace transforms and show how to accomodate some families of barriers which are curved with respect to forward prices. A section with concluding remarks ends the article.

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2 Canonical Transformations

Consider an asset A with value process A_t and let $F_t = F(A_t)$ be a shorthand notation for its forward price. Let $Z_t(T)$ be the price process for a zero coupon bond maturing at time T. Under the forward pricing measure $Q^{Z(T)}$ with $Z_t(T)$ as the numeraire asset, the process F_t is a martingale. We postulate the following stochastic model for F_t under the forward measure:

$$dF_t = \sigma(F_t)\zeta(t)dW_t. \tag{1}$$

It is legitimate to assume that $\zeta(t) = 1$, as the time change

$$t \to t' = \int^t \frac{ds}{\zeta(s)^2} \tag{2}$$

leads to this case. Let $P(F, t)$ be the price function for a derivative claim with a payoff $\phi(F)$ at maturity T and possibly with additional clauses forseeing the termination of the contract prior to maturity. The forward price at time t with delivery at maturity T is:

$$p(F_t, t) = \frac{P(F_t, t)}{Z_t(T)}. \tag{3}$$

Under the forward measure $Q^{Z(T)}$, the process $p_t = p(F_t, t)$ is a martingale and the expectation of the differential is zero

$$E_t^{Q(Z(T))}\big[dp_t\big] = 0. \tag{4}$$

Definition. A *canonical transformation* is specified by a constant δ, a function $X(F)$ and a numeraire asset G such that, under the pricing measure Q^G, the process

$$x_t = X(F_t), \tag{5}$$

has a constant drift δ.

The transformed process x_t obeys the following stochastic differential equation under the forward measure:

$$dx_t = \frac{1}{2}\sigma(F_t)^2 \frac{\partial^2 X}{\partial F^2}(F_t)dt + \sigma(F_t)\frac{\partial X}{\partial F}(F_t)dW_t \equiv \mu(x_t)dt + \nu(x_t)dW_t. \tag{6}$$

where

$$\mu(x) = \frac{1}{2}\sigma(F)^2 \frac{\partial^2 X(F)}{\partial F^2};\ \ \nu(x) = \sigma(F)\frac{\partial X(F)}{\partial F}. \tag{7}$$

The function $X(F_t)$ will be referred to as the *generating function* of the canonical transformation $x = X(F)$.

Let $G(x_t, t)$ be the pricing function for the derivative claim with numeraire asset G and let

$$g(x_t, t) = \frac{G(x_t, t)}{Z_t(T)} \tag{8}$$

be the corresponding forward price. Let $G(x, T) = g(x, T)$ be the payoff function at maturity T. Under the new measure Q^G, the ratio

$$h_t = h(x_t, t) \equiv \frac{p(F_t, t)}{g(x_t, t)} \tag{9}$$

follows a martingale process, i.e. $E^{Q(G)}\big[dh_t\big] = 0$. Here F_t is a function of x_t via Eq.(5). Notice that this representation is valid in general, under all probabilistic measures for the underlying stochastic process. Taking the stochastic differential of h_t, we find

$$0 = E^{Q(G)}\big[dh_t\big] = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}E^{Q(G)}\big[dx_t\big] + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}E^{Q(G)}\big[(dx_t)^2\big]. \tag{10}$$

Since by assumption x_t has constant drift δ under the pricing measure Q^G, we have that $E^{Q(G)}\big[dx_t\big] = \delta dt$. Moreover, from Eq.(6) one obtains $E^{Q(G)}\big[(dx_t)^2\big] = \nu(x)^2 dt$. Hence $h(x,t)$ solves the partial differential equation:

$$\frac{\partial h}{\partial t} + \delta\frac{\partial h}{\partial x} + \frac{\nu(x)^2}{2}\frac{\partial^2 h}{\partial x^2} = 0, \tag{11}$$

with final time condition

$$h(x,T) = \frac{p(F(x,T),T)}{g(x,T)} = \frac{\phi(F(x,T))}{g(x,T)} \tag{12}$$

The function $h(x,t)$ may also satisfy additional boundary conditions if there are clauses foreseeing the contract termination prior to maturity.

Theorem 1. *A transformation (X,G) is canonical if and only if*

$$\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial x} + \frac{\nu^2}{2}\frac{\partial^2 g}{\partial x^2} = 0. \tag{13}$$

and

$$(\mu - \delta)g + \nu^2\frac{\partial g}{\partial x} = 0 \tag{14}$$

Proof. Equation (13) follows from assuming that $g(x_t,t)$ is a martingale under the forward measure while using Eq.(6) and Ito's Lemma. This martingale property implies that g_t follows a replicable asset price process. Furthermore, if $p(F_t(x_t,t),t)$ is the pricing function of a contingent claim and $h(x_t,t)$ is the function defined as in (9), then the product $p(F_t(x_t,t),t) = h(x_t,t)g(x_t,t)$ must also be a martingale under the forward measure. Using Ito's Lemma again together with Eq.(6) within the expectation $E^{Q(Z)}[d(hg)] = 0$ gives the equation

$$\frac{\partial(hg)}{\partial t} + \mu(x)\frac{\partial(hg)}{\partial x} + \frac{\nu(x)^2}{2}\frac{\partial^2(hg)}{\partial x^2} = 0. \tag{15}$$

Carrying out the derivatives and collecting terms gives:

$$h\left[\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial x} + \frac{\nu^2}{2}\frac{\partial^2 g}{\partial x^2}\right] + g\left[\frac{\partial h}{\partial t} + \frac{\nu^2}{2}\frac{\partial^2 h}{\partial x^2}\right] + \left(\mu g + \nu^2\frac{\partial g}{\partial x}\right)\frac{\partial h}{\partial x} = 0. \tag{16}$$

Upon using Eqs.(11) and (13), we find

$$\left((\mu-\delta)g+\nu^2\frac{\partial g}{\partial x}\right)\frac{\partial h}{\partial x}=0. \tag{17}$$

Since this relation is obeyed for all choices of the pricing function $h(x)$, equation (14) follows.

3 Separation of Variables

We now specialize to the case where variables separate in the canonical transformation, i.e.

$$g(x,t)=\xi(x)\lambda(t), \qquad \mu=\mu(x), \qquad \nu=\nu(x). \tag{18}$$

Theorem 2. *Let $\nu(x)$ be a given volatility function and δ a constant drift. The drift $\mu(x)$ is given by*

$$\mu(x)=\nu(x)^2\frac{\psi'(x)}{\psi(x)}=\nu(x)^2\frac{d}{dx}\log\psi(x) \tag{19}$$

where $\psi(x)$, as given by

$$\psi(x)=\exp\left(\int^x\frac{\mu(s)}{\nu(s)^2}ds\right), \tag{20}$$

satisfies the one-dimensional Schrödinger type equation

$$-\psi''(x)+V(x)\psi(x)=0. \tag{21}$$

with potential

$$V(x)=\frac{1}{\nu(x)^2}\left[2\rho+\frac{\delta^2}{\nu(x)^2}-2\delta\frac{\nu'(x)}{\nu(x)}\right]. \tag{22}$$

Proof. Using the separation of variables ansatz $g(x,t)=\xi(x)\lambda(t)$, within the second condition for canonical transformations Eq.(14), one finds

$$\xi'(x)=-\frac{\mu(x)-\delta}{\nu(x)^2}\xi(x). \tag{23}$$

The first condition Eq.(13) gives

$$\frac{\dot{\lambda}}{\lambda} + \mu\frac{\xi'}{\xi} + \frac{\nu^2}{2}\frac{\xi''}{\xi} = 0. \tag{24}$$

Separation of variables x and t gives $\dot{\lambda}/\lambda = \rho$ (a constant independent of x and t), hence the function $\lambda(t)$ has the form

$$\lambda(t) = e^{\rho t}. \tag{25}$$

Making use of Eq.(23) and its derivative, we find

$$\nu^2\xi'' = -\mu'\xi - \big(2\nu\nu' + (\mu - \delta)\big)\xi'. \tag{26}$$

Hence the differential equation for $\xi(x)$ reduces to the following first order Riccati type of equation for the drift function $\tilde{\mu}(x) \equiv \mu(x) - \delta$:

$$\tilde{\mu}' + \frac{\tilde{\mu}^2}{\nu^2} - 2\left(\frac{\nu'}{\nu} - \frac{\delta}{\nu^2}\right)\tilde{\mu} - 2\rho = 0. \tag{27}$$

Introducing the new function defined by Eq.(20) and differentiating twice gives

$$\psi''(x) = \frac{1}{\nu^2}\left[\mu' + \frac{\mu^2}{\nu^2} - 2\frac{\nu'}{\nu}\mu\right]\psi(x). \tag{28}$$

Using $\mu = \tilde{\mu} + \delta$, the latter equation can be rewritten as

$$\psi''(x) = \frac{1}{\nu^2}\left[\tilde{\mu}' + \frac{\tilde{\mu}^2}{\nu^2} - 2\big(\frac{\nu'}{\nu} - \frac{\delta}{\nu^2}\big)\tilde{\mu} + \frac{\delta^2}{\nu^2} - 2\delta\frac{\nu'}{\nu}\right]\psi(x). \tag{29}$$

Upon using Eq.(27) gives Eq.(21) with $V(x)$ defined in Eq.(22). Once a solution to $\psi(x)$ is found, the drift is given by inverting Eq.(20), giving Eq.(19). We note also that the transformation

$$\tilde{\psi}(x) = \exp\left(-\delta\int^x \frac{ds}{\nu(s)^2}\right)\psi(x), \tag{30}$$

leads to another useful differential equation as alternative to Eq.(21):

$$\tilde{\psi}''(x) + \frac{2\delta}{\nu(x)^2}\tilde{\psi}'(x) - \frac{2\rho}{\nu(x)^2}\tilde{\psi}(x) = 0. \tag{31}$$

Solving for $\tilde{\psi}(x)$, the drift is obtained as

$$\mu(x) = \delta + \nu(x)^2 \frac{d}{dx} \log \tilde{\psi}(x). \tag{32}$$

4 Local Volatility by Quadratures

The second of the generating function equations (7) gives

$$\frac{\partial X(F)}{\partial F} = \frac{\nu(X(F))}{\sigma(F)} \tag{33}$$

which integrates to

$$\int^X \frac{dx}{\nu(x)} = \int^F \frac{df}{\sigma(f)}. \tag{34}$$

Hence the generating function can be derived from the two volatility functions $\nu(x)$ and $\sigma(F)$. The following result allows one to derive the function $\sigma(F)$ from the drift function $\mu(x)$ of a canonical transformation:

Theorem 3. *If the drift $\mu(x)$ is given, then the tranformation $x(F)$ and the volatility function $\sigma(F)$ can be obtained by quadratures.*

Proof. The equation

$$\mu(x) = \frac{\sigma(F)^2}{2} \frac{d}{dF} \left(\frac{\nu(x(F))}{\sigma(F)} \right) \bigg|_{F=F(x)}. \tag{35}$$

can be recast as follows:

$$\mu(x) = -\frac{\nu(x)^2}{2} \frac{d}{dF} \left(\frac{1}{x'(F)} \right), \tag{36}$$

where $x'(F) \equiv dx(F)/dF$. This is a consequence of

$$\frac{d}{dF} \left(\frac{\nu(x(F))}{\sigma(F)} \right) = x''(F), \qquad \text{and} \qquad \sigma(F(x)) = \frac{\nu(x)}{x'}. \tag{37}$$

This equation admits the following conserved quantity:

$$\mathcal{E}(x, x') = \log x' + \Phi(x) \tag{38}$$

where

$$\Phi(x) = -\int^x \frac{2\mu(y)dy}{\nu(y)^2}. \tag{39}$$

If E is an arbitrary integration constant, the solution with $\mathcal{E}(x, x') = E$ is given implicitly by

$$F = F(x) = c_0 + c_1 \int^x e^{\Phi(y)} dy \tag{40}$$

where $c_1 = e^{-E}$ and c_0 are constants. Notice that the function $F(x)$ is monotonically increasing, as expected from a coordinate change. The state-dependent volatility $\sigma(F)$ can hence be expressed as follows:

$$\sigma(F) = Ce^{\Phi(x(F))}\ \nu(x(F)), \tag{41}$$

for any constant C. The function $\Phi(x)$ above can be further recast in terms of $\psi(x)$ by using Eq.(19) into the above integral giving:

$$\Phi(x) = -2\int^x \frac{d}{dy} \log\psi(y)dy = -2\log\psi(x). \tag{42}$$

Moreover, the inverse coordinate transformation is

$$F = F(x) = c_0 + c_1 \int^x e^{\Phi(y)} dy = c_0 + c_1 \int^x \frac{dy}{\psi(y)^2} \tag{43}$$

and the volatility function is

$$\sigma(F) = Ce^{\Phi(x(F))}\nu(x(F)) = \frac{C\nu(x(F))}{\psi(x(F))^2}. \tag{44}$$

Note that in terms of the function $\tilde{\psi}(x)$ defined in (30) the above expressions are slightly more involved:

$$\Phi(x) = -2\log\tilde{\psi}(x) - 2\delta \int^x \frac{ds}{\nu(s)^2}, \tag{45}$$

$$F = F(x) = c_0 + c_1 \int^x \frac{e^{-2\delta \int^y \frac{ds}{\nu(s)^2}}}{\tilde{\psi}(y)^2} dy, \tag{46}$$

$$\sigma(F) = \frac{C\nu(x(F))e^{-2\delta \int^{x(F)} \frac{ds}{\nu(s)^2}}}{\tilde{\psi}(x(F))^2}. \tag{47}$$

5 The Wiener Family

If the volatility is constant, i.e. $\nu(x) = \nu$ (a constant), the potential in the associated Schrödinger equation is constant, i.e.

$$V(x) = \lambda_\nu^2 \equiv (2\rho + \delta^2/\nu^2)/\nu^2, \tag{48}$$

and $\psi(x)$ satisfies

$$\psi''(x) - \lambda_\nu^2 \psi(x) = 0. \tag{49}$$

One can distinguish between a number of special cases.

Case 1. In the special case of $\rho = \delta = \lambda_\nu = 0$, Eq.(49) admits a linear solution

$$\psi(x) = c_1 x + c_0 \tag{50}$$

and the drift is given by

$$\mu(x) = \frac{\nu^2}{x + c_2}, \tag{51}$$

where $c_2 = \frac{c_0}{c_1}$. The inverse coordinate transformation is

$$F(x) = c_2 + \int^x \frac{dy}{(c_0 + c_1 y)^2}, \tag{52}$$

hence $F(x)$ has the form

$$F(x) = c_2 + \frac{1}{(c_0 + c_1 x)}, \tag{53}$$

for constants c_0,c_1,c_2. The volatility function becomes

$$\sigma(F) = \frac{c}{\psi(x(F))^2} = \frac{c}{(c_1 x(F) + c_0)^2} = c(F - c_2)^2. \tag{54}$$

This solution where the volatility is a quadratic function with one root appears in [22]. Notice that in this case the solution blows up with finite probability, depending on the drift and the initial condition, as the underlying crosses the point $x_t = -\frac{c_0}{c_1}$.

Next, consider the case $\lambda_\nu^2 > 0$, i.e. $2\rho > \delta^2/\nu^2$. Then Eq.(21) admits the solutions

$$\psi(x) = C_+ e^{\lambda_\nu x} + C_- e^{-\lambda_\nu x} \tag{55}$$

where the $C_\pm$ are arbitrary coefficients. Using Eq.(19), the general drift model for which we have analytical solutions is therefore

$$\mu(x) = \nu^2 \lambda_\nu \frac{C_+ e^{\lambda_\nu x} - C_- e^{-\lambda_\nu x}}{C_+ e^{\lambda_\nu x} + C_- e^{-\lambda_\nu x}}. \tag{56}$$

Different choices of the constants $C_\pm$ give rise to various parameterized families for the drift. One can distinguish the following three cases:

Case 2. If one and only one of the constants $C_\pm$ vanishes, then

$$\mu(x) = \pm \nu^2 \lambda_\nu \tag{57}$$

In this case the drift $\mu(x)$ is constant, say $\mu(x) = \mu_0$. Since $\Phi(x) = -2(\mu_0/\nu^2)x$, the inverse transformation is given by

$$F = c_0 + c_1 \int^x e^{-\frac{2\mu_0}{\nu^2} y} dy = c_0 - \frac{c_1}{(2\mu_0/\nu^2)} e^{-\frac{2\mu_0}{\nu^2} x}, \tag{58}$$

and the volatility function is linear

$$\sigma(F) = c e^{-\frac{2\mu_0}{\nu^2} x} = c_2 (F - c_0) \tag{59}$$

where $c_2 = c(2\mu_0/\nu^2)/c_1$ is a constant. This solution appeared in [13]. Notice that in this particular case, the volatility function leading to an exactly solvable model does not depend on δ.

Case 3. The choice $C_+ = \frac{C}{2} e^a$ and $C_- = \frac{C}{2} e^{-a}$ leads to a drift function of the form

$$\mu(x) = \nu^2 \lambda_\nu \tanh(\lambda_\nu x + a) \tag{60}$$

where a, C are arbitrary constants. In this case $\psi(x) = C\cosh(\lambda_\nu x + a)$, and the function $\Phi(x)$ is given by

$$\Phi(x) = -2\log(C\cosh(\lambda_\nu x + a)) \tag{61}$$

and the inverse transformation is

$$F = F(x) = c_0 + \frac{c_1}{C^2}\int^x \frac{dy}{\cosh^2(\lambda_\nu y + a)} = c_0 + c_2\tanh(\lambda_\nu x + a), \tag{62}$$

where $c_2 = c_1/C^2\lambda_\nu$. The volatility has the form

$$\sigma(F) = \frac{c}{\cosh^2(\lambda_\nu x + a)}. \tag{63}$$

Combining Eqs.(62) and (63) while using the identity $\cosh^2 x - \sinh^2 x = 1$ and rearranging terms gives

$$\sigma(F) = \sigma_0 + \sigma_1 F + \sigma_2 F^2, \tag{64}$$

with constants $\sigma_0 = c(1-(c_0/c_2)^2)$, $\sigma_1 = 2c_0c/c_2^2$, $\sigma_2 = -c/c_2^2$. Factoring the form in Eq.(64), where all constants defined are real numbers, readily shows that this model corresponds to a quadratic volatility with two different real roots as introduced by Ingerssol in [10].

Case 4. The choice $C_+ = Ce^a/2$ and $C_- = -Ce^{-a}/2$ gives the drift in the form

$$\mu(x) = \nu^2\lambda_\nu\coth(\lambda_\nu x + a). \tag{65}$$

The algebra for this drift model is very similar to the above hyperbolic tangent case and also leads to the same quadratic volatility model with two real separate roots. In fact, $F(x) = c_0 - c_2\coth(\lambda_\nu x + a)$, $\sigma(F) = C/\sinh^2(\lambda_\nu x + a)$, giving Eq.(64) with constants $\sigma_0 = -C(1-(c_0/c_2)^2)$, $\sigma_1 = -2c_0C/c_2^2$, $\sigma_2 = C/c_2^2$.

The difference between the cases 3 and 4 is related to the initial condition for the corresponding stochastic differential equations for the “overlying” forward price process F_t. Given

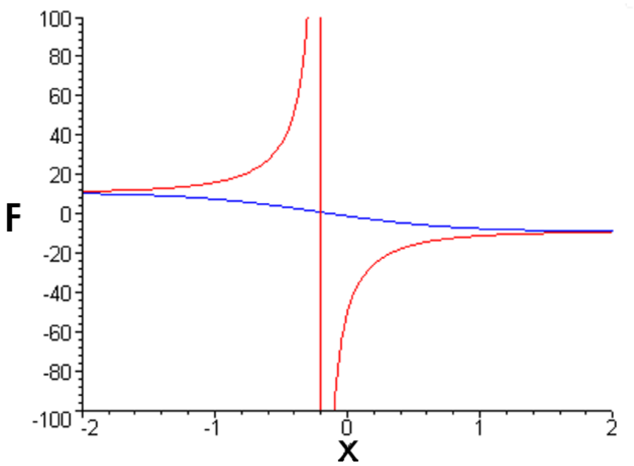

Figure 1: *The tanh and coth transformations compared.*

a quadratic volatility function $\sigma(F) = \sigma_0 + \sigma_1 F + \sigma_2 F^2$ with two different roots, if F_t exceeds the threshold $\bar{F} = c_0 - c_2$ at some point in time, then a solution can be reduced to Brownian motion via the hyperbolic cotangent substitution. Due to the divergence of this transformation as $x \rightarrow -a/\lambda_\nu$, a singularity develops with finite probability, depending on the drift and the initial condition. If on the other hand F_t is below the threshold $\bar{F}$ at some point in time, then the appropriate transformation is given by the hyperbolic tangent and the solution for the forward price will stay bounded in the interval $[0, \bar{F}]$ for all times.

Assume now that $\lambda_\nu^2 < 0$, i.e. $2\rho < \delta^2/\nu^2$. In this case λ_ν is purely imaginary. Defining the real parameter $\tilde{\lambda} = \mid \lambda_\nu \mid = \sqrt{\mid 2\rho + \delta^2/\nu^2 \mid} / \mid \nu \mid$, Eq.(21) admits the following solutions

$$\psi(x) = A \sin(\tilde{\lambda} x) + B \cos(\tilde{\lambda} x). \tag{66}$$

Using Eq.(19), the drift is given by

$$\mu(x) = \nu^2 \lambda_\nu \frac{A\cos(\tilde{\lambda}x) - B\sin(\tilde{\lambda}x)}{A\sin(\tilde{\lambda}x) + B\cos(\tilde{\lambda}x)}, \tag{67}$$

where A,B are constants. This also gives rise to two more separate model cases for the drift as follows.

Case 5. Choosing $A = -C\sin\theta$ and $B = C\cos\theta$, for any real constants θ, C, the drift takes the form

$$\mu(x) = -\nu^2 \tilde{\lambda} \tan(\tilde{\lambda}x + \theta) \tag{68}$$

where $\psi(x) = C\cos(\tilde{\lambda}x + \theta)$. Now $\Phi(x) = -2\log(C\cos(\tilde{\lambda}x + \theta))$ and the inverse transformation is

$$F = F(x) = c_0 + \frac{c_1}{C^2}\int^x \frac{dy}{\sin^2(\tilde{\lambda}y + \theta)} = c_0 + c_2\tan(\tilde{\lambda}x + \theta), \tag{69}$$

where $c_2 = (c_1/\tilde{\lambda}C^2)$. The volatility has the form

$$\sigma(F) = \frac{c}{\cos^2(\tilde{\lambda}x + \theta)} = c\big(\tan^2(\tilde{\lambda}x + \theta) + 1\big). \tag{70}$$

Combining Eqs.(69) and (70) gives the quadratic volatility

$$\sigma(F) = \sigma_0 + \sigma_1 F + \sigma_2 F^2, \tag{71}$$

with constants $\sigma_0 = c(1 + (c_0/c_2)^2)$, $\sigma_1 = -2c_0c/c_2^2$, $\sigma_2 = c/c_2^2$. In this case the roots of $\sigma(F)$ correspond to a complex conjugate pair. Under this model, the solution develops a singularity in a finite time with finite probability, depending on the drift and the inital condition.

6 The Bessel Family

We now consider solutions for the above root model where $\nu(x) = 2\sqrt{x}$ in the general case that $\rho \neq 0$. This corresponds to the stochastic Bessel pricing process with

$$dx = \delta dt + 2\sqrt{x}dW. \tag{72}$$

We note also that the drift can take on values $\delta > 2$. In this case, the fundamental solution of the Fokker-Plank equation

$$u_t = 2(xu)_{xx} - \delta u_x, \tag{73}$$

where subscript x denotes the partial derivative with respect to x, can be expressed in terms of modified Bessel functions as follows:

$$u(x,t;x_0,0) = \frac{1}{2}\left(\frac{x}{x_0}\right)^{\frac{1}{2}(\frac{\delta}{2}-1)} \frac{e^{-(x+x_0)/2t}}{t} I_{\delta/2-1}\left(\frac{\sqrt{xx_0}}{t}\right). \tag{74}$$

This function satisfies Eq.(73) as can be verified by plugging the function into Eq.(73), and after some algebra gives the modified Bessel equation $z^2 I''_\nu(z) + zI'_\nu(z) - (z^2+\nu^2)I_\nu(z) = 0$, with order $\nu \equiv (\delta/2) - 1$ and $z \equiv \sqrt{xx_0}/t$. As a transition probability function in x space, the integral over all allowable space must be normalized to unity,

$$\int_0^\infty dx\, u(x,t;x_0,0) = 1, \tag{75}$$

and in the limit $t \to 0$ becomes the Dirac delta function

$$\lim_{t\to 0} u(x,t;x_0,0) = \delta(x-x_0). \tag{76}$$

Moreover, the function also satisfies the important continuity condition

$$\int_0^\infty dy\, u(x,t;y,0)\, u(y,t;x_0,0) = u(x,2t;x_0,0). \tag{77}$$

These properties follow readily from the following integral identities:

$$\int_0^\infty dx\, x^{\nu/2} \exp(-\alpha x) I_\nu(2\beta\sqrt{x}) = \beta^\nu \frac{\exp(\beta^2/\alpha)}{\alpha^{\nu+1}} \tag{78}$$

and

$$\int_0^\infty dx\; \exp(-\alpha x) I_\nu(2\beta\sqrt{x}) I_\nu(2\gamma\sqrt{x}) = \frac{1}{\alpha} I_\nu(\frac{2\beta\gamma}{\alpha}) \exp\big(\frac{\beta^2+\gamma^2}{\alpha}\big) \tag{79}$$

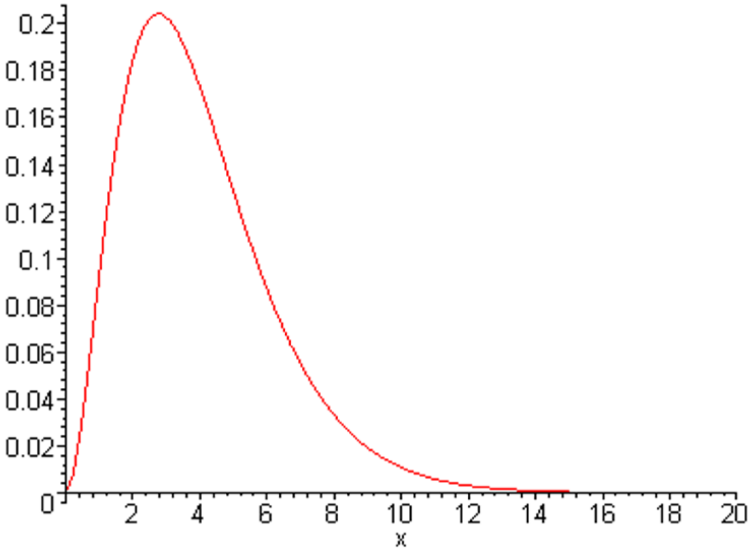

Figure 2: *The probability distribution in the Bessel model with* $\delta = 3$, $t = 1$ *and* $x_0 = 1$.

with order $Re\,\nu > -1$, and constants α, β, γ. Note that to obtain the limit for $t \longrightarrow 0$ one makes use of the asymptotic form: $I_\nu(z) \sim e^z/(2\pi z)$, as $z \longrightarrow \infty$.

For models that reduce to the Bessel process, the Shrödinger equation (21) takes the form

$$\psi''(x) + \left(\frac{(-\rho/2)}{x} - \frac{\delta^2/4 - \delta}{4x^2} \right) \psi(x) = 0. \tag{80}$$

We distinguish two cases for the most general solutions. If $\rho < 0$ then

$$\psi(x) = a_1 \sqrt{x} J_{\frac{\delta}{2}-1}(\sqrt{-2\rho x}) + a_2 \sqrt{x} Y_{\frac{\delta}{2}-1}(\sqrt{-2\rho x}), \tag{81}$$

while if $\rho > 0$,

$$\psi(x) = a_1 \sqrt{x} I_{\frac{\delta}{2}-1}(\sqrt{2\rho x}) + a_2 \sqrt{x} K_{\frac{\delta}{2}-1}(\sqrt{2\rho x}). \tag{82}$$

In both cases a_1 and a_2 are arbitrary constants. The case $\rho = 0$ is considered separately in the next section.

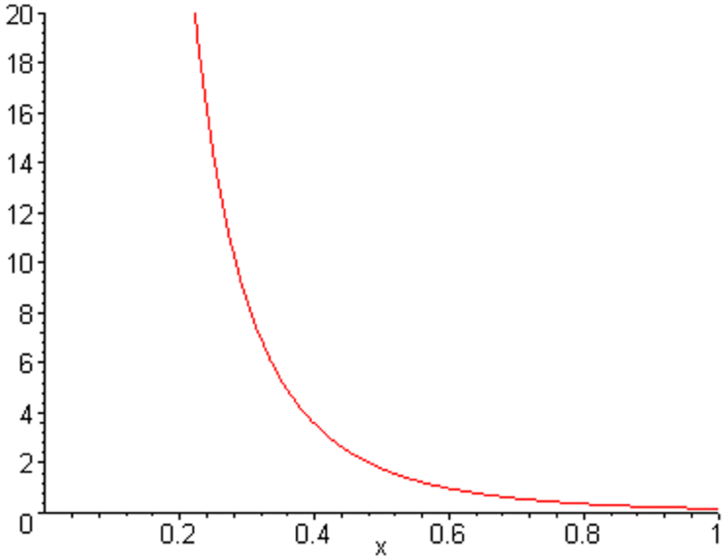

Figure 3: *The coordinate transformation in case $\psi(x)$ is a Bessel I function.*

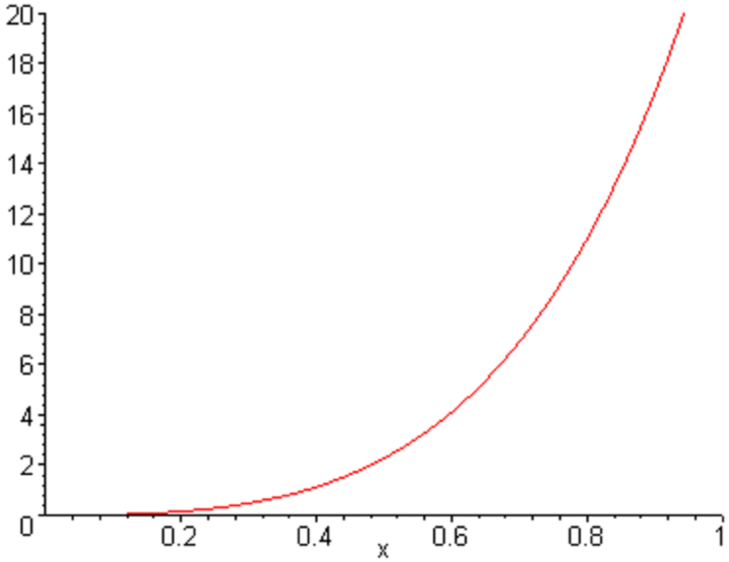

Figure 4: *The coordinate transformation in case $\psi(x)$ is a Bessel K function.*

The inverse coordinate transformation as given by the equation

$$F = F_0 + c_1 \int^x \frac{dy}{\psi(y)^2}. \tag{83}$$

exists on the entire half-line $x > 0$ only if the function ψ is expressed through the Bessel functions without zeroes, i.e. either through the K or through the I function. This criterion rules out the case $\rho < 0$, for which the solution blows up in finite time with finite probability, depending on the initial condition and the drift δ. The general form for $\psi(x)$ in Eq.(82) leads to analytic expressions for $F(x)$. This arises by consideration of the derivative identities:

$$\frac{d}{dz}\left(\frac{(1/a_2)I_\nu(z)}{a_1 I_\nu(z) + a_2 K_\nu(z)}\right) = \frac{1}{z[a_1 I_\nu(z) + a_2 K_\nu(z)]^2}, \tag{84}$$

for $a_2 \neq 0$, and

$$\frac{d}{dz}\left(\frac{-(1/a_1)K_\nu(z)}{a_1 I_\nu(z) + a_2 K_\nu(z)}\right) = \frac{1}{z[a_1 I_\nu(z) + a_2 K_\nu(z)]^2}, \tag{85}$$

for $a_1 \neq 0$. These identities are obtained via the Wronskian relation $I_\nu(z)K'_\nu(z) - K_\nu(z)I'_\nu(z) = -1/z$, for any order ν. Using the above identities and a change of integration variable gives:

$$F(x) = F_0 + \left(\frac{2c_1}{a_2}\right)\frac{1}{a_1 + a_2[K_{\delta/2-1}(\sqrt{2\rho x})/I_{\delta/2-1}(\sqrt{2\rho x})}, \tag{86}$$

for any positive $a_2 \neq 0$ and any $a_1 \geq 0$, or

$$F(x) = F_0 - \left(\frac{2c_1}{a_1}\right)\frac{1}{a_2 + a_1[I_{\delta/2-1}(\sqrt{2\rho x})/K_{\delta/2-1}(\sqrt{2\rho x})}, \tag{87}$$

for any positive $a_1 \neq 0$ and any $a_2 \geq 0$. Generally then we have

$$\xi(x) = \frac{x^{(\delta/2-1)/2}}{a_1 I_{\frac{\delta}{2}-1}(\sqrt{2\rho x}) + a_2 K_{\frac{\delta}{2}-1}(\sqrt{2\rho x})}. \tag{88}$$

We note that two sub-cases also arise when $a_1 = 0, a_2 = 1$, whereby

$$F(x) = F_0 + 2c_1 \frac{I_{\delta/2-1}(\sqrt{2\rho x})}{K_{\delta/2-1}(\sqrt{2\rho x})}, \qquad \xi(x) = \frac{x^{(\delta/2-1)/2}}{K_{\delta/2-1}(\sqrt{2\rho x})}, \tag{89}$$

and when $a_1 = 1, a_2 = 0$, whereby

$$F(x) = F_0 - 2c_1 \frac{K_{\delta/2-1}(\sqrt{2\rho x})}{I_{\delta/2-1}(\sqrt{2\rho x})}, \qquad \xi(x) = \frac{x^{(\delta/2-1)/2}}{I_{\delta/2-1}(\sqrt{2\rho x})}. \tag{90}$$

In all cases, with the choice $F_0 = 0$ and $c_1 > 0$, the mapping $F(x)$ is monotonous and maps the half-line $[0, \infty)$ to $[0, \infty)$. The difference is that in some cases $F(x)$ is increasing while in others the mapping is decreasing. The latter two equations define 2 families of pricing models with 4 parameters: ρ, δ, the scale parameter for the forward price c_1 and the scale parameter for the local volatility $\sigma(F)$. In the next section, we show that in the limit as $\rho \to 0$, the two scale parameters are equivalent and one recovers the *CEV* family of models.

Next, we assume F_0 and derive pricing formulas for a European style call option struck at K and with maturity T.

7 The Martingale Family

In this section we show how the drift generally reduces to quadratures for the case $\rho = 0$ and any non-constant volatility $\nu(x)$. Indeed, Eq.(31) gives

$$\frac{d}{dx} \log \tilde{\psi}'(x) = \frac{-2\delta}{\nu(x)^2}, \tag{91}$$

which is integrated to give

$$\tilde{\psi}'(x) = c_0 \exp\left(-2\delta \int^x \frac{ds}{\nu(s)^2} \right). \tag{92}$$

Another integration gives

$$\tilde{\psi}(x) = c_1 + c_0 \int^x dy \exp\left(-2\delta \int^y \frac{ds}{\nu(s)^2} \right). \tag{93}$$

or

$$\psi(x) = \exp\left(\delta \int^x \frac{ds}{\nu(s)^2} \right) \left(c_1 + c_0 \int^x dy \exp\left(-2\delta \int^y \frac{ds}{\nu(s)^2} \right) \right). \tag{94}$$

The calculation of the drift is therefore reduced to quadratures:

$$\mu(x) = \delta + \nu(x)^2 \frac{e^{-2\delta \int^x \frac{ds}{\nu(s)^2}}}{c + \int^x dy e^{-2\delta \int^y \frac{ds}{\nu(s)^2}}} \tag{95}$$

where c is a constant.

Example 1. Consider the root model for the volatility $\nu(x) = 2\sqrt{x}$. In this case we have the exact solutions

$$\tilde{\psi}(x) = c_1 + \frac{c_0}{1-(\delta/2)} x^{1-(\delta/2)} \tag{96}$$

for $\delta \neq 2$, and

$$\tilde{\psi}(x) = c_1 + c_0 \log x \tag{97}$$

for $\delta = 2$. The drift then takes the forms

$$\mu(x) = \delta + \frac{4 - 2\delta}{1 + c(1-\delta/2)x^{(\delta/2)-1}} \tag{98}$$

for $\delta \neq 2$, and

$$\mu(x) = \delta + \frac{4}{c + \log x} \tag{99}$$

for $\delta = 2$, where c is a constant. Specializing to the case $c_1 = 0$ and $\delta \neq 2$, we can assume without restricting generality that $\tilde{\psi}(x) = x^{1-(\delta/2)}$. In this case, the transformation is given by

$$F = F(x) = c_0' + c_1' \int^x \frac{y^{\delta-2}}{y^{\delta/2}} dy = c_0' - \frac{c_1'}{\delta/2 - 1} x^{-1+\delta/2}. \tag{100}$$

and the volatility function takes the form

$$\sigma(F) = 2C(x(F))^{-1/2} x((F))^{-1+\delta/2} = \sigma_0 (F + F_0)^{1+(\delta-2)^{-1}} \tag{101}$$

where F_0 and σ_0 are constants. Setting F_0, this model reduces to the CEV model in Cox and Ross [7] and Schröder [19].

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