STOCHASTIC MECHANICS AS A GAUGE THEORY

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Abstract. We show that non-relativistic Quantum Mechanics can be faithfully represented in terms of a classical diffusion process endowed with a gauge symmetry of group $\mathbb{Z}_4$. The representation is based on a quantization condition for the realized action along paths. A lattice regularization is introduced to make rigorous sense of the construction and then removed. Quantum mechanics is recovered in the continuum limit and the full $U(1)$ gauge group symmetry of electro-magnetism appears. Anti-particle representations emerge naturally, albeit the context is non-relativistic. Quantum density matrices are obtained by averaging classical probability distributions over phase-action variables. We find that quantum conditioning can be described in classical terms but not through the standard notion of sub $\sigma-$algebras. Delicate restrictions arise by the constraint that we are only interested in the algebra of gauge invariant random variables. We conclude that Quantum Mechanics is equivalent to a theory of gauge invariant classical stochastic processes we call Stochastic Mechanics.

Contents

The question of whether the Schrodinger equation can be interpreted as a classical diffusion attracted much attention since the discovery of quantum mechanics. Semiclassical expansions are introduced in (Weyl 1927) and (Wigner 1932) while a Hilbert space approach to quantum mechanics is introduced in (Koopman 1931) and (von Neumann 1932). These approaches are useful to bridge the gap between the classical and quantum formalisms but do not establish an equivalence. A more radical departure is in (Bohm 1952) and (Fenyes 1952), where new models are introduced which are not entirely consistent with standard Quantum Mechanics. This line of research was then greatly expanded upon in (Nelson 1967) in what became known as Nelson's Stochastic Mechanics. See also (de la Pena-Auerbach 1970), (Jammer 1974), (Guerra and Ruggiero 1973), (Bacciagaluppi 2005). The key difficulty stems from the inherent differences between Quantum and Classical Probability also revealed by Bell's inequalities regarding hidden variable theories, see (Bell 1966).

In this paper we introduce a purely classical representation for quantum mechanics which is entirely faithful. We do so by considering non-relativistic quantum mechanics but suspect that the construction is of general validity. The classical diffusion on which we base the analysis has a $\mathbb{Z}_4$ gauge symmetry and it is this symmetry which is responsible of the subtle differences between Quantum and Classical Probability theory. On the quantum side, this gauge symmetry is also related to the $U(1)$ gauge symmetry of electromagnetism. In Section 1 we summarize our results and in the following sections we give details.

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1. Quantization Condition

Consider a spinless particle of mass m and charge e in an external electromagnetic potential $(\overrightarrow{A}(x), \phi(x))$. To quantize the motion, consider the following diffusion equation:

$$d\overrightarrow{x}_t = -\frac{e}{cm}\overrightarrow{A}(\overrightarrow{x})dt + \sqrt{\frac{\hbar}{m}}d\overrightarrow{W}. \tag{1.1}$$

The action realized along a path is formally given by the process

$$S_t = \int_0^t \frac{m}{2}\left(\frac{d\overrightarrow{x}_t}{dt}\right)^2 dt + \frac{e}{c}\overrightarrow{A}(x)\cdot d\overrightarrow{x}_t - \phi(\overrightarrow{x}_t)dt. \tag{1.2}$$

The trouble with this equation is that, since the paths of the Wiener process are rough, the realized action is infinite. The singularities originate from the kinetic term only, while the other two terms are mathematically well defined as stochastic integrals. A regularization is thus required.

By using equation (1.1) we can rearrange this expression as follows to extract the first singular term:

$$S_t = \int_0^t \frac{\hbar}{2}\left(\frac{d\overrightarrow{W}_t}{dt}\right)^2 dt - V(\overrightarrow{x}_t)dt = \int_0^t \frac{m}{2}\left(\frac{d\overrightarrow{x}_t}{dt} + \frac{e}{cm}\overrightarrow{A}(\overrightarrow{x})\right)^2 dt - V(\overrightarrow{x}_t)dt. \tag{1.3}$$

where

$$V(x) = \frac{e^2}{2c^2m}\overrightarrow{A}(x)^2 dt + e\phi(\overrightarrow{x}_t). \tag{1.4}$$

To render all expressions finite and extract the singularities, we fix an elementary space scale $a > 0$ and an elementary time interval $\delta t > 0$. The strategy we follow is to discretize the diffusion process in (1.1) so that it evolves on $(a\mathbb{Z})^d$. Then, we regularize also the definition of kinetic terms in the action by setting

$$S_t = \sum_{j=0}^{\frac{t}{\delta t}} \frac{m}{2}\left(\frac{\overrightarrow{x}_{(j+1)\delta t} - \overrightarrow{x}_{j\delta t}}{\delta t} + \frac{e}{cm}\overrightarrow{A}(x)\right)^2 \delta t - \int_0^t V(x_t)dt. \tag{1.5}$$

As a criterion for the regularization scheme, we ask that the action be quantized according to

$$S_t = S_0 - \int_0^t V(x_t)dt + \hbar n_t + O\left(\frac{t\delta t\hbar^2}{a^4m^2}\right) \tag{1.6}$$

where n_t is an integer valued process. In Section 2, we show that it is possible to achieve this objective as long as one chooses the time discretization interval to be

$$\delta t = \frac{ma^2}{\hbar}. \tag{1.7}$$

Next, we notice that the joint process $(\overrightarrow{x}_t, n_t)$ is translation invariant in the n_t direction. This derives from the fact that the action has the form of an integral extended over a path: as time advances, the realized action is updated on the basis of the most recent changes and without memory effects. In (Albanese 2007*b*) and (Albanese 2007*a*) processes with similar symmetries are called Abelian as they are associated to a commutative operator algebra. Finally, we notice that the dynamics of n_t can further be restricted to $\mathbb{Z}_4$ by using mod 4 periodicity without compromising the Abelian character of the dynamics. This gives rise to a gauge symmetry for the classical process $(\overrightarrow{x}_t, n_t)$. Details are in Section 3, but we anticipate here some conclusions.

The joint kernel $\tilde{u}(\overrightarrow{x}, 0; \overrightarrow{x}', n; t)$ obviously preserves probability in the classical sense. However, as consequence of the Abelian symmetry of the action integral, Quantum Mechanics makes a wondrous appearance through the following equation:

$$e^{it\mathbb{H}}(\overrightarrow{x}, \overrightarrow{x}') = \exp\left((1-i)\frac{d\hbar^2 t}{ma^2} + 2K_0 t\right)\sum_{n=0}^{\infty} i^n \tilde{u}(\overrightarrow{x}, 0; \overrightarrow{x}', n; t). \tag{1.8}$$

On the right of this equation we see the classical probability distribution function for the joint process. Here, K_0 is an energy threshold we define more precisely below. On the left hand side, $\mathbb{H}$ is the quantum mechanical propagator for the Schrodinger operator

$$\mathbb{H} = -\frac{\hbar^2}{2m}\Delta_a + \frac{ie\hbar}{cm}\overrightarrow{A}(\overrightarrow{x}) \cdot \overrightarrow{\nabla}_a + V(x). \tag{1.9}$$

This formula depends on the lattice spacing a and the argument of the exponential factor in equation (1.8) diverges as $a \to 0$. However, the resulting Hamiltonian is well defined, its limit is regular and converges to the usual Hamiltonian for non-relativistic Quantum Mechanics.

These equations are intriguing as they unveil a mathematical relationship, but one more step is needed to arrive to a proper and physically grounded representation of Quantum Mechanics. Again, the existence of the $\mathbb{Z}_4$ gauge symmetry motivates us to revise the starting point and complicate a bit the classical model by creating two copies of it. As we explain in Section 4, we introduce a second independent joint process (ξ_t, ν_t). Here ξ_t satisfies also an equation of the form (1.1) except that it is driven by an independent Wiener process and ν_t is defined similarly to n_t except that it corresponds to the time-reversed process. We then look at the classical stochastic process with probability distribution function $\rho_c(\xi_t, \nu_t, x_t, n_t; t)$. Our main result can be expressed as follows:

Theorem 1. *The operator of matrix*

$$\rho_q(\xi_t, x_t; t) = \exp\left(2\frac{\hbar^2 t}{ma^2} + 4K_0 t\right) \sum_{n,\nu} i^{n-\nu} \rho_c(\xi_t, \nu_t, x_t, n_t; t) \tag{1.10}$$

solves the Quantum Mechanical equation for density matrices

$$\rho_q(t) = e^{-it\mathbb{H}} \rho(0) e^{it\mathbb{H}}. \tag{1.11}$$

Furthermore, all quantum mechanical density matrices can be written in the form (1.10) by averaging some classical joint probability density $\rho_c(\xi_t, \nu_t, x_t, n_t; t)$.

The proposed representation is thus completely equivalent to ordinary non-relativistic Quantum Mechanics. We apparently accomplished the feat of obtaining Quantum Mechanics out of a limiting procedure involving two independent copies of a classical stochastic process for two pairs of joint diffusions of a particle and the corresponding realized action on the particle world path. However, we cannot jump to the conclusion that there is a one-to-one correspondence between classical and quantum events. There are subtleties as the classical dynamics of quantized action is subject to a $\mathbb{Z}_4$ gauge symmetry. This symmetry is of pivotal importance in the construction and develops into the full fledged $U(1)$ gauge symmetry of quantum electro-dynamics in the quantum representation. It also affects the base mathematical structures and definition of the algebras of quantum events. A measurement apparatus built with physical matter is inescapably subject to the same $\mathbb{Z}_4$ gauge symmetry that we leveraged on. Hence, physical observables which are measurable ought to be gauge invariant. One thus needs to consider the σ-algebra of classical events along with the algebra of random variables with are gauge invariant. This is a subtle restriction that invalidates the standard constructs in Classical Probability such as that of conditional probability. This is also at the origin of quantum coherence, quantum entanglement and other departures of Quantum Mechanics from the standard Classical Probability. But when gauge symmetries are accounted for, faithful mathematical equivalence results.

The classical representation is not only mathematically complete and faithful, it also contains a few extra bits of known Physics. By changing the factor i^n into i^{-n} in the right hand side of equation (1.8), we find another interesting equation

$$e^{it\mathbb{H}'}(\overrightarrow{x}, \overrightarrow{x}') = \exp\left((1-i)\frac{\hbar^2 t}{ma^2} + 2K_0 t\right) \sum_{n=0}^{\infty} i^{-n} \tilde{u}(\overrightarrow{x}, 0; \overrightarrow{x}', n; t). \tag{1.12}$$

where $\mathbb{H}'$ is the PT reversal of $\mathbb{H}$, i.e. the operator obtained by inverting both time and space coordinates. Equivalently, it is the charge conjugate operator associated to antiparticles, as seen in relativistic quantum theory. Similarly one can conceive of seas filled of particles and anti-particles along the same lines. We conclude that relativity is not responsible for the existence of antimatter, gauge symmetries are.

2. Lattice Regularization

Consider physical space discretized on the lattice $(a\mathbb{Z})^3$ and consider the process described by the Markov generator

$$\mathcal{L}(\vec{x};\vec{x}') = \frac{\hbar}{2m}\Delta_a(\vec{x};\vec{x}') + \frac{e}{cm}\vec{A}^-(\vec{x})\cdot\vec{\nabla}_a^+(\vec{x};\vec{x}') + \frac{e}{cm}\vec{A}^+(\vec{x})\cdot\vec{\nabla}_a^-(\vec{x};\vec{x}') \tag{2.1}$$

where

$$\Delta_a(\vec{x};\vec{x}') = \sum_{i=1}^{3}\frac{\delta(\vec{x}'-\vec{x}-a\vec{e}_i)+\delta(\vec{x}'-\vec{x}+a\vec{e}_i)-2\delta(\vec{x}'-\vec{x})}{a^2} \tag{2.2}$$

$$\nabla_a^{i+}(\vec{x};\vec{x}') = \frac{\delta(\vec{x}'-\vec{x}-a\vec{e}_i)-\delta(\vec{x}'-\vec{x})}{a}$$

$$\nabla_a^{i-}(\vec{x};\vec{x}') = \frac{\delta(\vec{x}'-\vec{x}+a\vec{e}_i)-\delta(\vec{x}'-\vec{x})}{a}. \tag{2.3}$$

We set

$$A_i^+(\vec{x}) = \max(\vec{A}^i(x),0) \quad \text{and} \quad A_i^-(\vec{x}) = \max(-\vec{A}^i(x),0) \tag{2.4}$$

Let us rearrange this formula as follows:

$$\begin{aligned}\mathcal{L}(\vec{x};\vec{x}') = \frac{\hbar}{2ma^2}\sum_{i=1}^{3}\Bigg[&\left(1+\frac{2ae}{c\hbar}A_i^-(\vec{x})\right)\delta(\vec{x}'-\vec{x}-a\vec{e}_i)\\ &+\left(1+\frac{2ae}{c\hbar}A_i^+(\vec{x})\right)\delta(\vec{x}'-\vec{x}+a\vec{e}_i)\\ &-\left(2+\frac{2ae}{c\hbar}A_i^-(\vec{x})+\frac{2ae}{c\hbar}A_i^+(\vec{x})\right)\delta(\vec{x}'-\vec{x})\Bigg].\end{aligned} \tag{2.5}$$

Here, δ is the function such that $\delta(0)=1$ and $\delta(\vec{x})=0$ if $x\neq 0$.

Next, let us introduce an integer valued process n_t such that the joint process (x_t,n_t) is defined by the following lifted generator:

$$\begin{aligned}\tilde{\mathcal{L}}(\vec{x},n;\vec{x}',n') =& \frac{\hbar}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(\delta(n'-n+1)+\frac{2ae}{c\hbar}A_i^-(\vec{x})\delta(n'-n)\right)\delta(\vec{x}'-\vec{x}-a\vec{e}_i)\\ &+\left(\delta(n'-n+1)+\frac{2ae}{c\hbar}A_i^+(\vec{x})\delta(n'-n)\right)\delta(\vec{x}'-\vec{x}+a\vec{e}_i)\\ &-\left(2+\frac{2ae}{c\hbar}A_i^-(\vec{x})+\frac{2ae}{c\hbar}A_i^+(\vec{x})\right)\delta(\vec{x}'-\vec{x})\delta(n'-n)\Bigg]\\ &+\frac{1}{2\hbar}V(\vec{x})\delta(\vec{x}'-\vec{x})(\delta(n'-n-1)-\delta(n'-n+1))\\ &+\frac{1}{\hbar}K_0(\delta(n'-n-1)+\delta(n'-n+1)-2\delta(n'-n))\delta(\vec{x}'-\vec{x}).\end{aligned} \tag{2.6}$$

Here, δ is the function such that $\delta(0)=1$ and $\delta(\vec{x})=0$ if $x\neq 0$. $\delta(n)=1$ if $n=0$ and zero otherwise. Furthermore, we assume that

$$|V(\vec{x})|\leq 2K_0 \quad \text{and} \quad K_0\leq\frac{\hbar^2}{a^2m}. \tag{2.7}$$

The first condition ensures that off-diagonal elements in the Markov generator stay positive and the upper bound on K_0 is required in an estimate below. As $a\downarrow 0$, the energy cutoff K_0 also diverges.

The elementary propagator over a time interval δt is given by

$$u_{\delta t}(\overrightarrow{x}, n; \overrightarrow{x}', n') = e^{\delta t \tilde{\mathcal{L}}}(\overrightarrow{x}, n; \overrightarrow{x}', n'). \tag{2.8}$$

We find

$$\begin{aligned} E_t\big[n_{t+\delta t} - n_t \big| n_t = n, \overrightarrow{x}_t = \overrightarrow{x}\big] &= \sum_{n'x'} u_{\delta t}(\overrightarrow{x}, n; \overrightarrow{x}', n')(n' - n) \\ &= \frac{3\hbar\delta t}{ma^2} - \frac{V(\overrightarrow{x})\delta t}{\hbar} + O\Big(\frac{\delta t^2 \hbar^2}{a^4 m^2}\Big) \end{aligned} \tag{2.9}$$

where we made use of the bound in (2.7). We also have that

$$\begin{aligned} \frac{1}{\hbar} E_t\Big[S_{t+\delta t} - S_{t+\delta t}\Big] &= \frac{\delta t}{\hbar} E_t\Big[\frac{m}{2}\Big(\frac{\overrightarrow{x}_{(j+1)\delta t} - \overrightarrow{x}_{j\delta t}}{\delta t} + \frac{e}{cm}\overrightarrow{A}(\overrightarrow{x})\Big)^2 - V(\overrightarrow{x})\Big| n_t = n, \overrightarrow{x}_t = \overrightarrow{x}\Big] \\ &= \frac{\delta t}{\hbar} \sum_{n'\overrightarrow{x}'} u_{\delta t}(\overrightarrow{x}, n; \overrightarrow{x}', n')\Big[\frac{m}{2}\Big(\frac{\overrightarrow{x}' - \overrightarrow{x}}{\delta t} + \frac{e}{cm}\overrightarrow{A}(\overrightarrow{x})\Big)^2 - V(\overrightarrow{x})\Big]\delta t \\ &= \frac{3}{2} - \frac{V(\overrightarrow{x})\delta t}{\hbar} + O\Big(\frac{\delta t^2\hbar^2}{a^4 m^2}\Big) \end{aligned} \tag{2.10}$$

Hence, if

$$\delta t = \frac{ma^2}{\hbar} \tag{2.11}$$

then the regularized version of the action is quantized in multiples of $\hbar$ in the sense of equation (1.6). If the limit as $a \downarrow 0$ is taken while holding the ratio $\frac{a^2}{\delta t}$ fixed, the action will stay correctly quantized.

3. The Joint Process for the Position and the Realized Action

Notice that the lifted generator in (2.6) is defined in such a way that the Markov generator is invariant under translations in the direction of n. Using the terminology in (Albanese 2007*b*) and (Albanese 2007*a*), the pair $(\overrightarrow{x}_t, S_t)$ is an example of Abelian process. We can thus single out a sector with respect to the translation symmetry in the n direction.

Before proceeding, let us also notice that the lifted generator can be interpreted as describing a dynamics on the reduced configuration space $(a\mathbb{Z})^d \times \mathbb{Z}_4$ by identifying values of n which are equal modulo 4. As far as this generator is concerned, we could even restrict n to $\mathbb{Z}_3$ but then we would not recover Quantum Mechanics. The additional symmetry that $\mathbb{Z}_4$ has with respect to $\mathbb{Z}_3$ appears to be essential in the argument below.

Consider the partial Fourier transform operator of kernel

$$\mathcal{F}(\overrightarrow{x}, p; \overrightarrow{x}', n) = e^{-ipn}\delta_{xx'} \tag{3.1}$$

where $p \in [0, 2\pi)$. Partial Fourier transforms in the n variable are a block-diagonalizing transformation for the lifted generator. Let us introduce the operator $\hat{\mathcal{L}}$ such that

$$\big(\mathcal{F}\tilde{\mathcal{L}}\mathcal{F}^{-1}\big)(\overrightarrow{x}, p; \overrightarrow{x}', p') = \hat{\mathcal{L}}(\overrightarrow{x}, \overrightarrow{x}'; p)\delta_{pp'}, \tag{3.2}$$

i.e.

$$\begin{aligned}\hat{\mathcal{L}}(\overrightarrow{x},\overrightarrow{x}';p) =& \frac{\hbar}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(e^{-ip}+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x}-a\overrightarrow{e}_i)\\ &+\left(e^{-ip}+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x}+a\overrightarrow{e}_i)\\ &-\left(2+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{x})+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{x})\right)\delta(\overrightarrow{x}'-\overrightarrow{x})\Bigg]\\ &+\frac{1}{2\hbar}V(\overrightarrow{x})\delta(\overrightarrow{x}'-\overrightarrow{x})(e^{ip}-e^{-ip})\\ &+\frac{1}{\hbar}K_0(e^{ip}+e^{-ip}-2)\delta(\overrightarrow{x}'-\overrightarrow{x}).\end{aligned} \tag{3.3}$$

The sector with $p=\frac{\pi}{2}$ is special because in this case we recover the quantum mechanics Hamiltonian, i.e.

$$\hat{\mathcal{L}}\left(\overrightarrow{x};\overrightarrow{x}';\frac{\pi}{2}\right)=\frac{i}{\hbar}\mathbb{H}(\overrightarrow{x};\overrightarrow{x}')-\left[(1-i)\frac{d\hbar}{ma^2}+\frac{2K_0}{\hbar}\right]\delta(\overrightarrow{x}'-\overrightarrow{x}). \tag{3.4}$$

where

$$\mathbb{H}(\overrightarrow{x};\overrightarrow{x}')=-\frac{\hbar^2}{2m}\Delta_a(\overrightarrow{x};\overrightarrow{x}')+\frac{ie\hbar}{cm}\overrightarrow{A}(\overrightarrow{x})\cdot\overrightarrow{\nabla}_a(\overrightarrow{x};\overrightarrow{x}')+V(x)\delta(\overrightarrow{x}'-\overrightarrow{x}). \tag{3.5}$$

Let us notice that the kernel of the stochastic process on the principal bundle $(a\mathbb{Z})^3\times\mathbb{Z}_4$ is given by

$$\tilde{u}(t)=e^{t\tilde{\mathcal{L}}}=\mathcal{F}^{-1}\exp\left(t\mathcal{F}\tilde{\mathcal{L}}\mathcal{F}^{-1}\right)\mathcal{F}=\mathcal{F}^{-1}e^{t\hat{\mathcal{L}}}\mathcal{F}. \tag{3.6}$$

Similarly, we have that

$$\hat{u}\left(\overrightarrow{x},\overrightarrow{x}';\frac{\pi}{2},t\right)=\exp\left(t\hat{\mathcal{L}}\left(\frac{\pi}{2}\right)\right)(\overrightarrow{x},\overrightarrow{x}')=\sum_{n=0}^{\infty}i^n\tilde{u}(\overrightarrow{x},0;\overrightarrow{x}',n;t)=\exp\left(it\mathbb{H}-(1-i)\frac{d\hbar^2t}{ma^2}\right)(\overrightarrow{x},\overrightarrow{x}') \tag{3.7}$$

More explicitly, the quantum mechanical kernel can be reconstructed from the probabilistic kernel as follows:

$$e^{it\mathbb{H}}(\overrightarrow{x},\overrightarrow{x}')=\exp\left((1-i)\frac{d\hbar^2t}{ma^2}+2K_0t\right)\sum_{n=0}^{\infty}i^ne^{t\tilde{\mathcal{L}}}(\overrightarrow{x},0;\overrightarrow{x}',n;t). \tag{3.8}$$

This is an intriguing formula as it relates a quantum mechanical observable to a classical diffusion kernel.

4. Density Matrices

We have seen that because of the Abelian character of the process $(\overrightarrow{x}_t,n_t)$, i.e. of the translation invariance with respect to the n coordinate, the Fourier momentum p conjugate to n is conserved. It is natural to make the hypothesis that only observables whose expectation is indifferent to the action of the $\mathbb{Z}_4$ gauge group are physically measurable. In fact, a measurement apparatus itself would have to be a physical system subject to the same symmetry. The problem with this loose statement is that one cannot make it mathematically precise if one uses a representation with only one classical particle. To make things work, we need to complicate the construction and assign to each single particle in the quantum representation two particles which evolves independently in the classical representation.

As a full classical description of a quantum particle we take a quadruplet $(\overrightarrow{\xi}_t,\nu_t;\overrightarrow{x}_t,n_t)$. Each pair $(\overrightarrow{x}_t,n_t)$ and $(\overrightarrow{\xi}_t,\nu_t)$ is postulated to evolve independently. The first pair evolves according to the generator in (2.6). For the second pair, we introduce a conjugate process whose dynamics also provides a different lifting of the same base process in (1.1), given by the following generator:

$$
\begin{aligned}
\tilde{\mathcal{G}}(\overrightarrow{\xi},\nu;\overrightarrow{\xi}',\nu') =& \frac{\hbar}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(\delta(\nu'-\nu-1)+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{\xi})\delta(\nu'-\nu)\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}-a\overrightarrow{e}_i)\\
&+\left(\delta(\nu'-\nu-1)+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{\xi})\delta(\nu'-\nu)\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}+a\overrightarrow{e}_i)\\
&-\left(2+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{\xi})+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{\xi})\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi})\delta(\nu'-\nu)\Bigg]\\
&+\frac{1}{2\hbar}V(\overrightarrow{\xi})\delta(\overrightarrow{\xi}'-\overrightarrow{\xi})(\delta(\nu'-\nu+1)-\delta(\nu'-\nu-1))\\
&+\frac{1}{\hbar}K_0(\delta(\nu'-\nu-1)+\delta(\nu'-\nu+1)-2\delta(\nu'-\nu))\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}).
\end{aligned}
\tag{4.1}
$$

The Fourier transformed kernel is

$$
\begin{aligned}
\hat{\mathcal{G}}(\overrightarrow{\xi},\overrightarrow{\xi}';p) =& \frac{\hbar}{2ma^2}\sum_{i=1}^{3}\Bigg[\left(e^{ip}+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{\xi})\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}-a\overrightarrow{e}_i)\\
&+\left(e^{ip}+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{\xi})\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}+a\overrightarrow{e}_i)\\
&-\left(2+\frac{2ae}{c\hbar}A_i^-(\overrightarrow{\xi})+\frac{2ae}{c\hbar}A_i^+(\overrightarrow{\xi})\right)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi})\Bigg]\\
&+\frac{1}{2\hbar}V(\overrightarrow{\xi})\delta(\overrightarrow{\xi}'-\overrightarrow{\xi})(e^{-ip}-e^{ip})\\
&+\frac{1}{\hbar}K_0(e^{ip}+e^{-ip}-2)\delta(\overrightarrow{\xi}'-\overrightarrow{\xi}).
\end{aligned}
\tag{4.2}
$$

Notice that this Fourier transformed generator is the complex conjugate of the generator in (3.3). Since the Hamiltonian is self-adjoint, we have that

$$
\hat{\mathcal{G}}\left(\overrightarrow{\xi},\overrightarrow{\xi}';\frac{\pi}{2}\right) = -i\mathbb{H}(\overrightarrow{\xi}',\overrightarrow{\xi}),
\tag{4.3}
$$

where $\mathbb{H}$ is the Hamiltonian operator in (3.5).

Consider the joint classical probability density for the process $(\overrightarrow{\xi}_t,\nu_t;\overrightarrow{x}_t,n_t)$ and write it as follows:

$$
\rho_c(\overrightarrow{\xi},\nu;\overrightarrow{x},n;t).
\tag{4.4}
$$

Also form the following reduced matrix

$$
\rho_q(\overrightarrow{\xi},\overrightarrow{x};t) = \exp\left(2\frac{d\hbar^2 t}{ma^2}+4K_0t\right)\sum_{n,\nu=0}^{3} i^{n-\nu}\rho_c(\overrightarrow{\xi},\nu;\overrightarrow{x},n;t).
\tag{4.5}
$$

By the results in the previous section, we know that

$$
\rho_q(t) = e^{-it\mathbb{H}}\rho(0)e^{it\mathbb{H}}.
\tag{4.6}
$$

This means that the operator $\rho(t)$ is a properly defined quantum mechanical density matrix. We thus proved that a quantum mechanical density matrix is given by a phase average of the classical diffusion times an exponential factor of time.

The density matrix we constructed may or may not correspond to a pure state. The derivation holds in general. A pure state is obtained if there exists a wave-function $\psi(x,t)$ such that

$$
\rho_q(\overrightarrow{\xi},\overrightarrow{x};t) = \psi(\xi,t)^*\psi(x,t).
\tag{4.7}
$$

A generic classical joint density will give a valid generic quantum matrix once phase averaging is carried out.

A physical observable in the classical description is given by a random variable of the form

$$F_c(\overrightarrow{\xi}, \nu; \overrightarrow{x}, n) = i^{n-\nu} F_q(\overrightarrow{x}; \overrightarrow{\xi}). \tag{4.8}$$

Let us denote this correspondence with

$$\pi F_q = F_c. \tag{4.9}$$

In this case, the classical expectation of the classical observable coincides with the quantum expectation, i.e.

$$\sum_{\xi,x} \rho_q(\overrightarrow{\xi}; \overrightarrow{x}) F_q(\overrightarrow{\xi}; \overrightarrow{x}) = \exp\left(2\frac{d\hbar^2 t}{ma^2} + 4K_0 t\right) \sum_{\xi,x,n,\nu} \rho_c(\overrightarrow{\xi}, \nu; \overrightarrow{x}, n)(\pi F_q)(\overrightarrow{\xi}, \nu; \overrightarrow{x}, n). \tag{4.10}$$

Classical dynamics is recovered in the limit as the mass $m \to \infty$ or as $\hbar \to 0$. In this limit, phase averaging speed becomes infinitely fast and the quantum density matrix $\rho_q(\overrightarrow{\xi}; \overrightarrow{x})$ is localized along the line $x = y$. In this case, the standard stationary phase argument shows that trajectories minimize the classical action.

5. Classical and Quantum Conditioning

In classical probability one follows Kolmogorov in giving a structure of σ-algebra to event space. The situation here is rather simple as we are making use of a space discretization and could safely assume also a finite volume cutoff. So one can safely say that the set of all possible events Σ_c is given by the set of all finite sets of quadruples $(\xi, \nu, x, n) \in (a\mathbb{Z})^d \times \mathbb{Z}_4 \times (a\mathbb{Z})^d \times \mathbb{Z}_4$ contained within a given large box. The point is that the set of all random variables on Σ_c is too large as not all functions correspond to physical observables. What we need is to restrict random variables to only those of the form $F_q = \pi F_c$ where $F_c(\xi, x)$ is a function independent of ν and n. The problematic part of this is that this reduction cannot be accomplished by reducing the sigma-algebra to a quantum subalgebra and declaring that quantum observables are the ones measurable with respect to the sub-algebra. This standard way of phrasing conditioning fails here as there is no sub-algebra such that all the corresponding measurable functions are precisely those of the form $F_q = \pi F_c$ while non-gauge invariant functions are non-measurable. This gauge symmetry is the reason why quantum conditioning is not equivalent to classical conditioning.

When conditioning to a quantum event, we consider two quantum observables $F_q(\xi, x)$ and $G_q(\xi, x)$ and need to give meaning to the conditional expectation

$$E\big[F|G = g\big] \tag{5.1}$$

We proceed in such a way that if the construction is applied to ordinary probabilistic conditioning and both the quantum density function and G are diagonal, then the result is the same. Namely, we interpret the equation $G = g$ we start by an unconditioned density matrix ρ_q and we construct the "closest" density matrix ρ_q^g such that

$$\mathcal{T}r(\rho_q^g G^k) = g^k \tag{5.2}$$

for all $k \geq 0$. The point is to define what we mean by "closest". If for simplicity g is a discrete eigenvalue of G and P_g is the corresponding eigenspace, then a good definition is to set

$$\rho_q^g = \frac{P_g \rho_q P_g}{\mathcal{T}r(P_g \rho_q P_g)}. \tag{5.3}$$

This definition uniquely specifies ρ_q^g. Out of the quantum density matrix we can reconstruct many conditional classical densities. The lack of uniqueness of the construction is however immaterial since we are interested in taking expectations only of classical observables which are gauge invariant.

All the standard quantum mechanics now follows. In particular, it is clear that two quantum observables $F(\xi, x)$ and $G(\xi, x)$ are simultaneously measurable only if they commute when interpreted as matrices.

6. Conclusions

We have defined a representation for non-relativistic quantum mechanics which is entirely equivalent to the standard theory but which is expressed in terms of a classical diffusion.

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