DYNAMIC CREDIT CORRELATION MODELING

C. ALBANESE, O. CHEN, A. DALESSANDRO, AND A. VIDLER

ABSTRACT. As the market for credit baskets and single tranche bespoke CDOs keeps growing very rapidly, we witness a lively debate about the merits and shortcomings of the base correlation model, which is currently the recognized market standard. Difficulties such as the lack of arbitrage-freedom and the witnessed impossibility to calibrate in some market situations are motivations to research a different standard for mark-to-market and risk management. To contribute to this ongoing debate, we describe here a new modeling framework based on a structural, bottom-up approach. Points of interest for this model are that it can be made consistent with many data sources such as rating transition probabilities, historical default probabilities, single name credit spread curves and equilibrium recovery swap rates. Remarkably enough, we find that the model can be calibrated simultaneously to synchronous datasets for the iTraxx and CDX term structures for tranche spreads across the entire capital structures, including the index basis, and for maturities up to 10 years. The model makes use of an innovative mathematical framework based on spectral analysis and provides numerically noiseless spreads and hedge ratios. As far as execution times are concerned, the model is at least as fast if not faster than the most simplified analytic versions of the base correlation model.

1. INTRODUCTION

The base correlation framework owes its popularity to the fact that it lends itself most directly to analytic closed form solutions and calibration methods based on straightforward interpolations. Unfortunately it is unclear whether there is a simple set of conditions guaranteeing arbitrage freedom. Instead, one can construct pathological examples of baskets with arbitrage free tranche spread specifications which cannot be calibrated. One can also show that the model is able to produce tranche spreads which are not arbitrage free. See (Livsey and Schloegl 2006). More elaborate multifactor copula models which are arbitrage free have been proposed but need to be tackled with Montecarlo simulation methods and are numerically more intensive. See (Laurent and Gregory 2003, McGinty *et al.* 2004) for reviews and references. The slow convergence of Monte Carlo methods especially for the purpose of evaluating hedge ratios requires substantial computational resources even in these very streamlined copula based frameworks. On the other hand, hedge ratios resulting from the analytically solvable base correlation model are often rather

---

Date: first version: `27 February 2005`, this version: `May 12, 2006.`

Antonio Dalessandro and Alicia Vidler worked on this project as part of the requirements toward their doctoral studies at Imperial College London and the opinions herein expressed are their own and not necessarily endorsed in any way by their employers. We are grateful to Brynjar Arnarrsson, Peter Carr, Mark Davis, William Dellal, Stefano Galiani, Darius Gatarek, Helyette Geman, Manfred Gilli, Andrew Rennie, Pedro Tavares, Stathis Tompaidis and Manlio Trovato for discussions. We are also grateful to two referees for useful comments. All remaining errors are our own.

crude and of problematic interpretation. Copula models are not associated with a Markovian dynamics for spreads and it is thus not possible to use copulas to price trading strategies and payoffs crucially dependent on the dynamics of the credit process. Examples are strategies in the index basis and equity tranchelets and prices of CDO tranche options and hybrids derivatives.

We find that a fully structural and economically detailed approach is not only required to understand the complexities of CDOs, but is also numerically viable. While in reduced form models in general and copula models in particular, obligors can either be operational or in a state of default and defaults occur unexpectedly, we choose to work with a structural model giving a very detailed description of the credit process. In this paper, we associate to each reference name a variable representing a measure of distance to default and varying continuously in the interval $[0, 1]$. Here 0 corresponds to the state of default and 1 is the unattainable limit of perfect credit-worthiness. Within this rich framework, we model an arbitrage free dynamics for spreads along the process for forward loss distributions of portfolios. We also succeed to estimate the market prices of credit risk and of correlation risk, thus reconciling the historical measure with single name CDS spreads and index tranche spreads. Remarkably enough, we find that our model can be brought into agreement with synchronous datasets for the iTraxx and CDX simultaneously. As byproducts, our model gives rise to a methodology for extrapolating CDS curves, assessing liquidity spreads, pricing credit baskets in terms of the basis [defined as the different between the average single name CDS spread and the basket spread], interpolate and extrapolate the term structure of tranche spreads consistently from 1 to 10 years. Having calibrated the model to the main indices, this framework results in a platform to consistently price and risk-manage index tranches with bespoke attachment-detachment points and concentration bespoke CDOs. As we show in the previous paper (Albanese and Chen 2005*d*), the model can also be completed to a fully structural description spanning the entire corporate capital structure and can be used to price intrinsic credit-equity hybrids such as equity default swaps and convertible bonds.

This article is based on research that developed over a time span of 6 years whose progress is documented in a series of earlier papers, see (Albanese *et al.* 2003, Albanese and Chen 2005*a*, Albanese and Chen 2005*b*, Albanese and Chen 2005*c*). In these articles, we discussed single name, through-the-cycle calibration for a class of credit barrier models. To tackle the problem of credit baskets and CDOs, the formalism is here restructured and made more powerful. Although conceptually very similar, technical differences are such that knowledge of the previous papers is not needed and the present article is essentially self-contained. One of the main difference with the previous papers is that the approach described here is non-parametric and does not make use of analytic solvability in terms of special functions. The mathematical apparatus based on spectral theory is very much the same, but the technical difficulties associated and restrictions associated to the use of special functions have been bypassed by the use of numerical linear algebra routines that allow one to evaluate the spectrum and eigenfunctions of the relevant operators. We find that the use of numerical linear algebra makes the model more flexible and essentially non-parametrically while retaining all the advantages of closed form solvability. This article extends the methodology in the previous papers also in a second direction as it includes correlation modeling and arrives to a pricing framework for credit baskets and synthetic CDOs. To achieve this objective, we introduce here a novel technique to model dynamic correlations by conditioning a very high dimensional continuous time lattice model to macro scenarios for the credit cycle. Both advances on the modeling side have numerous other applications to multi-factor pricing models and different asset classes that will be the subject of other papers.

A recent stream of literature worth mentioning has tackled the challenge of modeling credit portfolios dynamically. A richer dynamic framework is provided by the so called "top-down" frameworks recently proposed in Bennani (2005), Sidenius *et al.* (2005) and Schönbucher (2005). These approaches differ from ours as they are based on a reduced form dynamics whereby defaults occur unexpectedly. Furthermore, these models ignore the identity of the portfolio constituents and attempt to model directly the forward loss distribution. Arbitrage freedom in our model is automatic as we work in the minimal Markov filtration. In top-down models instead, one works in an enlarged filtration which must be dynamically constrained to ensure absence of arbitrage, in a spirit similar to what is done in the context of LIBOR market models. In our construction, we can also incorporate at no numerical cost detailed single name information such as industry a regional specificities. Our main modeling variable can be interpreted as a distance to default ranging between 0 and 1, whereby 0 corresponds to the state of default and 1 is an unattainable upper limit corresponding to perfect credit-worthiness. This variable undergoes a local Levy process with jumps and state dependent local volatility. The resulting Markov framework automatically ensures arbitrage freedom by construction, without the need of deriving and enforcing drift restrictions.

The calibration scheme for CDO tranches involves three stages. In the first, we work under the statistical measure (denoted with $\mathcal{P}$). Model parameters are estimated to achieve consistency with average historical rating transitions and default probabilities. Individual obligors share the same process specification but differ in the starting point for credit quality and in the term structure of recovery rates. We model the market price of credit risk by a change of drift designed in such a way to achieve agreement in aggregate with market spread rates. In the third stage, we overlay a final measure change in such a way to achieve consistency with the spreads of liquid tranches of index CDOs used as benchmarks, such as for instance iTraxx and CDX, thus estimating the market price of correlation risk. As a byproduct, we also obtain a valuation for the index basis, i.e. the difference between the average single name CDS spread of the constituents and the basket credit spread. This article is dedicated precisely to the description of an estimation methodology for the third and final step. We find that in the post summer 2005 period, a high precision fit for both the CDO tranches and the index basis can be obtained for both the iTraxx and the CDX indices. Remarkably, we find that this can be achieved by using the very same model parameters for both indices, at the condition that the datasets are synchronous. The only difference between the two indices is that we postulate different recovery rates for the two basket to reflect regional specificities. Namely, we assume that average equilibrium recovery swap spread rates are equal to 55% for the CDX and to 30% for the iTraxx. Since our model is capable of interpreting both leading standard portfolios simultaneously, we believe it is a strong candidate as a model for bespoke synthetic CDOs.

The calibration procedure for the correlation portion of the model is subdivided into two sub-steps. In the first, marginal transition probabilities describing the individual obligor dynamics are kept unchanged, implementing thus what could be called a "dynamic copula". The non-recombining binomial tree describing the index process is linked to credit quality ξ by means of a response function $\beta(\xi)$ and by specifying a set of correlation couplings between jump amplitudes. These are determined at first in a way to obtain an approximate agreement with observed tranche spreads across seniority levels. In a next sub-step, we then overlay a minimalistic measure change with the objective of achieving a perfect match with the CDO tranche spreads quoted in the market while leaving unaltered single name CDS curves. This second measure change can be interpreted as accounting for the market price of correlation risk. We obtain multi-modal loss distributions, with the number of modes growing with the time horizon and finally merging into a bell shaped profile reminiscent of a Gaussian distribution on time horizons of 7-10 years. We also observe that convergence to the Gaussian limit is much faster for high yield and even cross-over portfolios than it is for high grade portfolios.

This calibration procedure reflects a through-the-cycle approach to modeling and estimation. A point in time approach incorporating explicitly information regarding the economic status and the single name outlook would arguably be more effective, especially for concentration bespoke CDOs. However, this would require basing the estimation to a more detailed multi-year dataset of rating transition probabilities resolved by year, industry and jurisdiction. We are currently working in this direction and can anticipate that, although more data intensive, the mathematical framework outlined in this paper extends and is numerically just as efficient. The contribution of this paper is to demonstrate that within a simpler through-the-cycle but otherwise highly detailed bottom-up approach, one can indeed reconstruct consistently the term structure of tranche spreads across different benchmark indices by means of a high dimensional lattice model.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In section 3 we discuss in more detail the calibration procedure. In section 4 we describe the general methodology to condition the continuous time lattice to a factor process. Section 5 concludes with a discussion of examples.

2. Modeling distance to default

In this section we recast the credit barrier models discussed in (Albanese *et al.* 2003, Albanese and Chen 2005*b*, Albanese and Chen 2005*c*, Albanese and Chen 2005*a*) in a non-parametric framework, whereby tractability comes from the use of numerical linear algebra as opposed to coming from the analytical tractability of special functions.

2.1. **The underling diffusion process.** The first building block of our construction is a Markov chain process x_t on the lattice $\Omega = \{0, h, ..., hN\} \subset [0,1]$ where N is a positive integer and $h = 1/N$. In the case of a discretized diffusion with state dependent drift and volatility, the infinitesimal generator $\mathcal{L}$, of the process x_t is a tridiagonal matrix $\mathcal{L}_x$ computed on the basis of a drift function $\mu(\xi)$ and a volatility function $\sigma(\xi)$. Here, $\xi \in [0,1]$ is the distance to default variable and the tridiagonal generator is computed so that, on each lattice point x we have that:

$$\begin{array}{rl} \sum_y \mathcal{L}(x,y)(y-x) & = \mu(hx) \\ \sum_y \mathcal{L}(x,y)(y-x)^2 & = \sigma(hx)^2 \\ \sum_y \mathcal{L}(x,y) & = 0 \end{array}$$

We make use of two drift functions: $\mu_{\mathcal{P}}(\xi)$ and $\mu_{\mathcal{Q}}(\xi)$, one defining the $\mathcal{P}$ or statistical measure and the latter modeling the $\mathcal{Q}$ or pricing measure. We postulate that the only difference between the $\mathcal{P}$ and the $\mathcal{Q}$ measure lies in the specification of these two drift functions. Correspondingly, we use the subscripts $\mathcal{P}$ and $\mathcal{Q}$ to identify the Markov generator and transition probabilities under the corresponding measure. Whenever the subscripts are omitted as here below, formulas apply to both the $\mathcal{P}$ and the $\mathcal{Q}$ measure.

To manipulate the Markov generator by means of functional calculus, the first step is to diagonalize it. Let λ_n be the eigenvalues of the operator $\mathcal{L}$ and let $u_n(x)$ be the right eigenvectors, so that

$$\mathcal{L}u_n = \lambda_n u_n.$$

In most cases, Markov generators admit a complete set of eigenvectors. Although there are exceptions where diagonalization is not possible and one can reduce the operator at most to a non trivial Jordan form with non-zero off-diagonal elements, these exceptional situations occur very rarely both in a measure theoretic sense, as exceptions span a set of zero measure, and in a topological sense as their complement is dense in the space of all generators. In practical terms, this implies that non-diagonalizable operators arise very rarely if at all in practice and whenever they do, a professional numerical diagonalization algorithm would detect the problem and a small perturbation of the model parameters would rectify the problem. To carry out numerical diagonalization, we find that the function `dgeev` in the public domain package LAPACK is quite suitable.

In the following, we thus assume that the operator $\mathcal{L}$ admits a complete set of eigenvectors. In this case, we can form the matrix U whose columns are given by the eigenvectors $u_n(x)$ and write

$$\mathcal{L} = U\Lambda U^{-1}. \tag{1}$$

We denote with V the operator U^{-1} and with $v_n(x)$ its row vectors.

Key to our constructions is the remark that if the matrix operator $\mathcal{L}$ is diagonalisable we can apply an arbitrary function F to it by means of the following formula:

$$F(\mathcal{L}) = UF(\Lambda)U^{-1} \tag{2}$$

This formula is at the basis of the so-called "functional calculus". As Ito's formula regarding functions of stochastic processes is central in the stochastic analysis for diffusion processes, functional calculus for Markov generators plays a pivotal role in our framework for stochastic volatility models. This formula has several applications. An immediate one allows us to express the pricing kernel $u(x,t;y,t')$ of the process as follows:

$$u(x,t;y,t') = (e^{(t'-t)\mathcal{L}})(x,y) = \sum_n e^{\lambda_n(t'-t)} u_n(x)v_n(y). \tag{3}$$

2.2. **Introducing jumps.** At this stage of the construction we add jumps. Jumps are ubiquitous in credit models and we find that a jump component is necessary in order to reconcile observed default probabilities with credit migration probabilities. Within our framework, adding jumps involves marginal additional complexities from the numerical viewpoint.

To reflect asymmetries in the jump intensities, we model separately up and down jumps. A particularly interesting class of jump processes is associated to stochastic time changes given by non-decreasing processes T_t with independent increments. These time changes are known as Bochner subordinators and are characterized by a Bochner function $\phi(\lambda)$ such that

$$E_0\left[e^{-\lambda T_t}\right] = e^{-\phi(\lambda)t} \tag{4}$$

The variance gamma process, see (Madan and Seneta 1990, Madan *et al.* 1998), makes use of a gamma distributed time-change whose Bernstein function is

$$\phi(\lambda) = \frac{\mu^2}{\nu}\log\left(1+\lambda\frac{\nu}{\mu}\right) \tag{5}$$

where μ is the mean rate and ν is the variance rate of the variance gamma process. The generator of the jump process can be expressed using functional calculus as the operator $-\phi(-\mathcal{L})$. To produce asymmetric jumps, we specify the two parameters in (5) differently for the up and down jumps and compute two Markov generators separately

$$\mathcal{L}_\pm = -\phi_\pm(-\mathcal{L}) = -U\phi(-\Lambda_\pm)V \tag{6}$$

where

$$\phi_\pm(\lambda) = \frac{\mu_\pm^2}{\nu_\pm}\log\left(1+\lambda\frac{\nu_\pm}{\mu_\pm}\right) \tag{7}$$

The new generator for our process with asymmetric jumps is obtained by combining the two generators above

$$\mathcal{L} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mathcal{L}_-(2,1) & d(2,2) & \mathcal{L}_+(2,3) & \cdots & \mathcal{L}_+(2,n) \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathcal{L}_-(n-1,1) & \mathcal{L}_-(n-1,2) & \cdots & d(n-1,n-1) & \mathcal{L}_+(n-1,n) \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Here the element of the diagonal are chosen in such a way to satisfy the condition of probability conservation

$$d(x,x) = -\sum_{y\neq x}\mathcal{L}(x,y) \tag{8}$$

Also notice that we set the matrix elements at the upper and lower boundary to zero in such a way to implement absorbing boundary conditions at both boundaries. Only the boundary condition at the lower boundary is financially motivated, while the reason why we impose absorbtion also at the upper boundary is to have a conservative measure of the hitting probability of the upper boundary. Since the volatility decays linearly to zero there, this probability is small and the boundary condition allows us to estimate the error and keep it under control.

3. Estimation and calibration

The calibration procedure of the distance to default processes involves three separate and sequential steps. Firstly we estimate the $\mathcal{P}$ measure in such a way to ensure agreement with rating transition probabilities and historical probabilities of default. Secondly we estimate the market price of credit risk from the term structure of quoted single name CDS spreads. The third and last stage consists in the evaluation of the pricing measure $\mathcal{Q}^c$ for CDO tranches which includes an adjustment for the market price of correlation risk.

3.1. **Estimation under the historical measure $\mathcal{P}$.** We first estimate the process for distance to default x_t with respect to the statistical measure $\mathcal{P}$ by matching transition probabilities over one year and default probabilities over time horizons of 1, 3 and 5 years.
A credit rating system consists of a number K of different classes. In the case of the extended system by Moody's, $K = 18$ and the ratings are:

$$\{0, 1, \dots, 17\} \leftrightarrow \{\text{Default}, \text{Caa}, \text{B3}, \text{Ba3}, \text{Ba2}, \dots, \text{Aa3}, \text{Aa2}, \text{Aa1}, \text{Aaa}\}$$

We subdivide the nodes of the lattice Ω into K subintervals of adjacent nodes:

$$I_i = [x_{i-1}, ...x_i] \tag{9}$$

where $0 = x_0 < x_1 < ... < x_K = N$ and $\#(x_i - x_{i-1}) = \frac{N}{K}$, for $i = 1, ..., K$. The interval I_i corresponds to the i-th rating class. If a process is in I_i at time t, then it is said to have a credit rating of i. For all i, $\bar{x}_i \in I_i$ denotes the initial node. The conditional transition probability $\tilde{p}_{ij}(t)$ that an obligor with a given initial rating i at time 0 will have a rating j at a later time $t > 0$ can be estimated by matching it with historical averages provided by credit assessment institutions. For our purposes, we model this quantity as follows:

$$\tilde{p}_{ij}(t) = \sum_{y=a_{j-1}}^{a_j - 1} u_P(0, \bar{x}_i; t, y).$$

where $\bar{x}_i$ is a point in the interval I_i which represents the barycenter of the population density in that credit class and is part of the model specification. For simplicity's sake, we take $\bar{x}_i$ to be the midpoint of the interval I_i.

Absorption into the state $x = 0$ is interpreted as the occurrence of default. The probability that starting from the initial rating i and reaching a state of default by time t is given by

$$\tilde{p}_i^D(t) = u_{\mathcal{P}}(0, \bar{x}_i; t, 0).$$

The model under $\mathcal{P}$ is characterized by a drift function $\mu_{\mathcal{P}}(\xi)$, a volatility function $\sigma(\xi)$ and jump intensities. The first two functions are graphed in Fig. 2 and Fig. 1, while the variance rates we estimated are $\nu_+ = 7.5, \nu_- = 4$.

3.2. **Calibration under $\mathcal{Q}$.** Risk neutralization is defined by changing the drift function $\mu_{\mathcal{P}}(\xi)$ into $\mu_{\mathcal{Q}}(\xi)$, while leaving everything else unaltered.

The new drift is chosen in such a way to fit spread curves. Term structures of probability of default for each rating class are given by

$$\tilde{q}_i^D(t) = u_{\mathcal{Q}}(0, \bar{x}_i; t, 0). \tag{10}$$

We use CDS spreads for 125 names in the Dow Jones CDX index and 125 more names for the European iTraxx index. We consider two synchronous datasets for

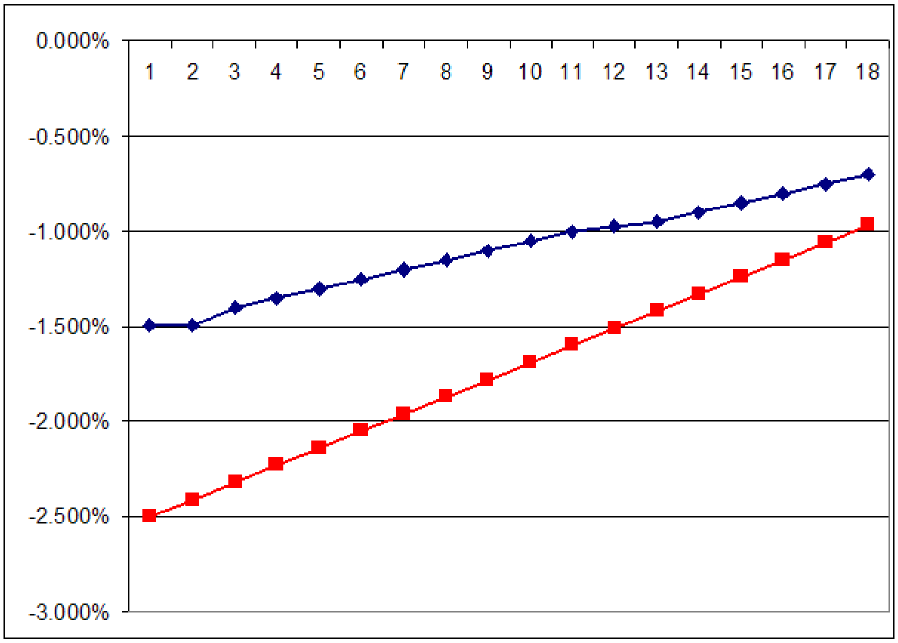

FIGURE 1. Local drifts $\mu_P(\xi)$ and $\mu_Q(\xi)$ for distance to default under the P and the Q measure, respectively.

both indices corresponding to October 31st 2005 and November 31st 2005. The datasets provide CDS spreads at maturities: 6m, 1y, 2y, 3y, 5y, 7y, 10y. When estimating, we insist that the recovery swap term structure be constant and match a given one for each name. Not having a reliable quote for recovery swap spreads, we postulate that equilibrium recovery swap rates are of 55% for the CDX names and 30% for the iTraxx names. This choice reflects empirical evidence on geographical dependencies of recovery rates. Given a choice for the drift under $\mathcal{Q}$, one can estimate the current distance to default for each name to achieve the best fit with the postulated term structure of recovery swap spreads. The drift term is specified in such a way to minimize the least square deviation of recovery swap spreads across all names in the two portfolios. For simplicity, we choose the same credit risk adjustment for both index portfolios.

The estimated function $\mu_Q(\xi)$ is graphed in Fig. 1. An example of term structures of equilibrium recovery swap rates for the CDX components is given in Fig. 4.

4. MODELING CORRELATIONS BY CONDITIONING

Having characterized the process for credit quality x_t and identified starting points for each individual process, the next step is to introduce correlations. To avoid the curse of dimensionality, we do so by conditioning to credit cycle scenarios, thus introducing a correlation structure among the credit quality processes. We illustrate this estimation step by considering calibration two synchronous datasets for CDX and iTraxx tranches for October and November 2005.

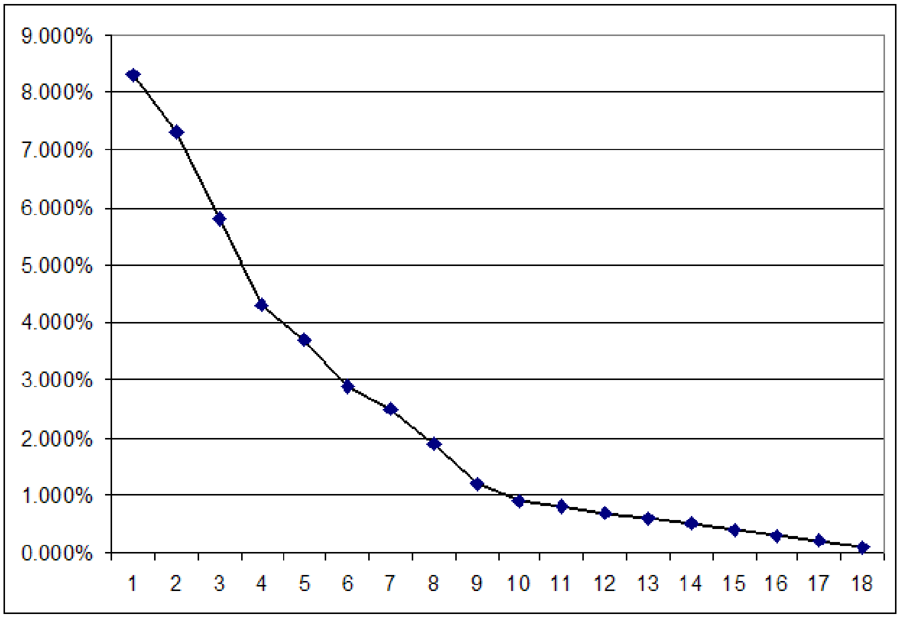

FIGURE 2. Local volatility $\sigma(\xi)$ for distance to default

4.1. **Dynamic Copula.** The credit cycle is modeled by means of a non-recombining lattice of the structure sketched in Fig. 20. The weights $w_1, w_2, \ldots$ are assigned to each scenario's path as in Fig. 21.

The underlying index variable is allowed to take up two values on each period Δt. An upturn corresponds to a period of spread tightening while a downturn corresponds to a period of spread widening. In our example, we chose the time step to be $\Delta t = 1y$ and find that this choice is sufficient to provide great flexibility in the tuning of the correlation structure.

To explain our methodology to introduce correlations, we consider first a simple case whereby the model is characterized by a pair of complementary transition probabilities $w, (1-w) \in [0,1]$ at each node, which we assume constant. In order to condition the continuous time lattice corresponding to a given credit quality process to the credit cycle index variable we introduce the notion of *local beta* given by function $\beta(\xi)$ which provides the corresponding sensitivity. The limiting cases of $\beta(\xi) = 0$ and $\beta(\xi) = 1$ correspond to zero and full correlation between a name with a given credit quality $hx \in [0,1]$ and the cycle variable. The local beta function, $\beta(\xi)$, is plotted in Fig.3.

Along the path of each given scenario on the tree, the unconditional kernel of the credit quality process is replaced by conditional transition probabilities defined as follows:

$$u^{\pm}_{w,\beta}(t,x;t+\Delta t,y) = (1-\beta(hx))u_0(t,x;t+\Delta t,y) + \beta(hx)u^{\pm}_1(t,x;t+\Delta t,y)$$

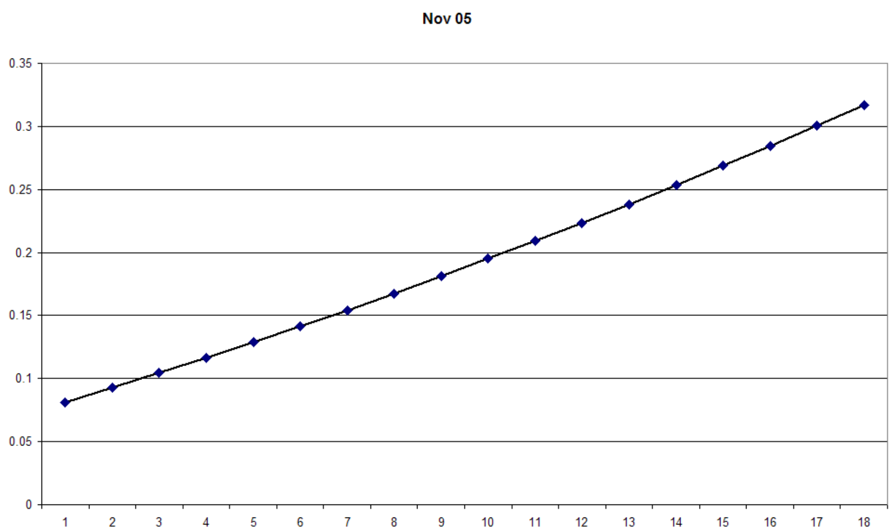

FIGURE 3. Specifications for the function $\beta(x)$ for both the October and November 2005 datasets.

Here $u_0 = u$ is the unconditional kernel and corresponds to a zero $\beta(hx)$. In the opposite case of $\beta(hx) = 1$, conditional kernel $u_1^{\pm}(x, y)$ has the following form:

$$u_1^+(x,y) = \frac{1}{1-w}\begin{cases} u(x,y) & \text{if} \quad y > m(w,x) \\ u(x,y) - u_m^-(x,y) & \text{if} \quad y = m(w,x) \\ u(x,y) = 0 & \text{if} \quad y < m(w,x) \end{cases} \tag{11}$$

and

$$u_1^-(x,y) = \frac{1}{w}\begin{cases} 0 & \text{if} \quad y > m(w,x) \\ u_m^-(x,y) & \text{if} \quad y = m(w,x) \\ u(x,y) & \text{if} \quad y < m(w,x). \end{cases} \tag{12}$$

where

$$m(w,x) = \inf\left\{k = 0, ...N|\sum_{y<k} u(x,y) \leq w\right\}. \tag{13}$$

More generally the formulation of the conditional probability kernel is as following:

$$u_\beta^+ = \begin{cases} (1-\beta(x))u(x,y) & \text{if} \quad y < m(w,x) \\ (1-\beta(x))u(x,y) + \frac{1}{1-w}\beta(x)u_m^+ & \text{if} \quad y = m(w,x) \\ (1-\beta(x))u(x,y) + \frac{1}{1-w}\beta(x)u(x,y) & \text{if} \quad y > m(w,x). \end{cases} \tag{14}$$

and

$$u_\beta^- = \begin{cases} (1-\beta(x))u(x,y) + \frac{1}{w}\beta(x)u(x,y) & \text{if} \quad y < m(w,x) \\ (1-\beta(x))u(x,y) + \frac{1}{w}\beta(x)u_m^- & \text{if} \quad y = m(w,x) \\ (1-\beta(x))u(x,y) & \text{if} \quad y > m(w,x) \end{cases} \tag{15}$$

where

$$u_m^- = \begin{cases} w & \text{if} \quad m = 0 \\ w - \sum_{y \le m} u(x,y) & \text{if} \quad m > 0. \end{cases} \tag{16}$$

and $u_m^+ = u(x,y) - u_m^-$.
Notice that, for all specifications of $\beta(\xi)$ and $w \in [0,1]$, we have that

$$u(x,y) = w u_\beta^-(x,y) + (1-w) u_\beta^+(x,y). \tag{17}$$

4.2. **Calibration under $\mathcal{Q}^c$: conditional probabilities of default.** Conditioning is achieved by forming a weighted sum over all paths in the event tree. On a given path, we use u_β^- for a spread-widening period scenario and u_β^+ for a spread-tightening period. The weight of a path is the product of a number of factors w equal to the number of spread-widening periods and a number of factors $(1-w)$ for each one of the spread-tightening periods, see Fig. 20. With this method, marginal probabilities are kept unchanged while correlations are induced on the single name processes. More specifically, one can price all credit sensitive instruments specified with the given names one can first evaluate the conditional prices P_Γ by means of the following multiperiod kernel:

$$e^{(t_i - t_{i-1})\mathcal{L}_{\gamma_i}} \cdot \ldots \cdot e^{(t_n - t_{n-1})\mathcal{L}_{\gamma_i}}.$$

where $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ runs over the sets of conditional paths due to the scenario of the index. The (unconditional) price is then given by:

$$P = \sum_\Gamma w^{n_-(\Gamma)} (1-w)^{n_+(\Gamma)} P_\Gamma.$$

This construction can be generalized. Consider a number $M > 1$ of percentile levels $0 < w_1 < ... < w_M < 1$ and let $q_i \in [0,1], i = 1...M$ be a corresponding set of probabilities summing up to one, i.e. $\sum_i q_i = 1$. Then we can set

$$u^\pm_{\vec{w},\beta}(t,x;t+\Delta t,y) = \sum_{i=1}^M q_i u^\pm_{w_i,\beta}(t,x;t+\Delta t,y). \tag{18}$$

The formulas above extend also to this case as long as one replaces the weight w with the average weight $\sum_i q_i w_i$. The choices we make to calibrate to iTraxx and CDX tranche spreads prevailing in October and November 2005 are given in Fig. 21. These two weight assignments are the only way the calibrated model differs between the two different dates. As one can see, the differences are rather minor.

Modeling correlations is key to pricing basket credit derivatives. As pointed out from Walker (2004), buyers and sellers of basket credit derivatives have a wide range of arbitrage-free prices to choose from, and it is the market that determines, both in principle and in practice, a definite price. In our framework, tranches of varying seniority are priced by calibrating the local beta function $\beta(\xi)$. We choose for $\beta(\xi)$ an increasing function of the credit quality to reflect the fact that higher rated names are more correlated to the general credit cycle than lower rated ones. This method allows us to avoid Monte Carlo methods entirely in the pricing of CDO's and in the calculation of the various hedge ratios.

We observe that our lattice model performs quite efficiently notwithstanding the fact that we are working in dimension 125. We separate the numerical analysis in two different steps. In the first we go through all names and generate conditional lattices. We choose $\Delta t = 1y$ and a time horizon of $5y$, so that we obtain a total of

32 scenarios. As a second step, we generate the loss distribution function for CDOs along with computing the present value of the premium legs for each tranche over the given time horizon, see also (Merino and Nyfeler 2002, Debuysscher *et al.* 2003). To price the premium leg of the basket, i.e. the 0-100 tranche, we also need to compute the forward distribution giving the total number of names that default within any given time horizon.

The model can be calibrated by adjusting the function $\beta(\xi), \xi \in [0,1]$, the thresholds $w_i, i = 1, ...M$ and the corresponding probabilities q_i. As an example, we show calibration results that simultaneously match the CDX and iTraxx tranche spreads of the most liquid tranches to within an average error of 2% on two dates: Oct 31 and Nov 31 2005.

As an example of term structure of tranche spreads produced by the model, we give in Figs. 5, 6 the November results. Figs. 5 and 5 give the term structure of index basis for the same two datasets. The loss distributions we obtain for the two November datasets are graphed in Figs. 5-5. Finally, in Fig.16, 17 , 18 and 19 we show the hedge ratios for the various CDO tranches obtained by slightly perturbing the CDS curves for each reference name and regressing quadratically versus the basket spread.

We notice that hedge ratios appear as numerically stable and noiseless. The tranches of shorter maturities have higher deltas in the more subordinated portions of the capital structure, while tranches of longer maturity are more sensitive to mezzanine risk. The gammas have alternating sign and sum up approximately to zero when averaged across the capital structure. this reflects the fact that, in a first approximation, holding protection in all tranches is roughly equivalent to holding index protection, and the letter position has no convexity. This is however only an approximate relation and there the actual average of gamma ratios deviates from zero.

5. Conclusions

We propose a novel approach to dynamic credit correlation modeling that is based on continuous time lattice models correlated by conditioning to a non-recombining tree. The model describes not only default events but also rating transitions and spread dynamics, while single name marginals are preserved. We find that tranche spreads for the CDX and iTraxx index portfolios in the period subsequent to the summer 2005 can be calibrated simultaneously.

References

Albanese, C. and O. Chen (2005*a*). Credit barrier models in a discrete framework. *Contemporary Mathematics 351, Mathematical Finance* pp. 1–11.

Albanese, C. and O. Chen (2005*b*). Discrete credit barrier models. *Quantitative Finance* **5**, 247–256.

Albanese, C. and O. Chen (2005*c*). Implied migration rates from credit barrier model. *The Journal of Banking and Finance, to appear.*

Albanese, C. and O. Chen (2005*d*). Pricing equity default swaps. *Risk Magazine* **18(6)**, 83–87.

Albanese, C., J. Campolieti, O. Chen and A. Zavidonov (2003). Credit barrier models. *Risk* **16**(6), 109–113.

Bennani, N. (2005). The forward loss model: A dynamic term structure approach for the pricing of portfolio credit derivatives.

Debuysscher, A., M. Szegö, M. Freydefront and H. Tabe (2003). The fourier transform method - technical document. *Moody's Investor Service, Working Paper* pp. 1–40.

Laurent, J.P. and J. Gregory (2003). Basket default swaps, cdo's and factor copulas.
Livsey, M. and L. Schloegl (2006). Recovery rate assumptions and no-arbitrage in the tranche market.
Madan, D. B. and E. Seneta (1990). The variance gamma (V.G.) model for share market returns. *Journal of Business* **63(4)**, 511–524.
Madan, D.B., P.P. Carr and E.C. Chang (1998). The variance gamma process and option pricing. *European Finance Review* **79**, 79–105.
McGinty, L., R. Ahluwalia, E. Beinstein and M. Watts (2004). Credit correlation: A guide. *JP-Morgan, Research Paper* pp. 1–36.
Merino, S. and M. Nyfeler (2002). Calculating portfolio loss. *Risk,* **August**, 82–86.
Schönbucher, P. J. (2005). Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives.
Sidenius, J., V. Piterbarg and L. Andersen (2005). A new framework for dynamic credit portfolio loss modelling.
Walker, M.B. (2004). Risk-neutral correlation in the pricing and hedging of basket credit derivatives. *Journal of Credit Risk, to appear* pp. 1–6.

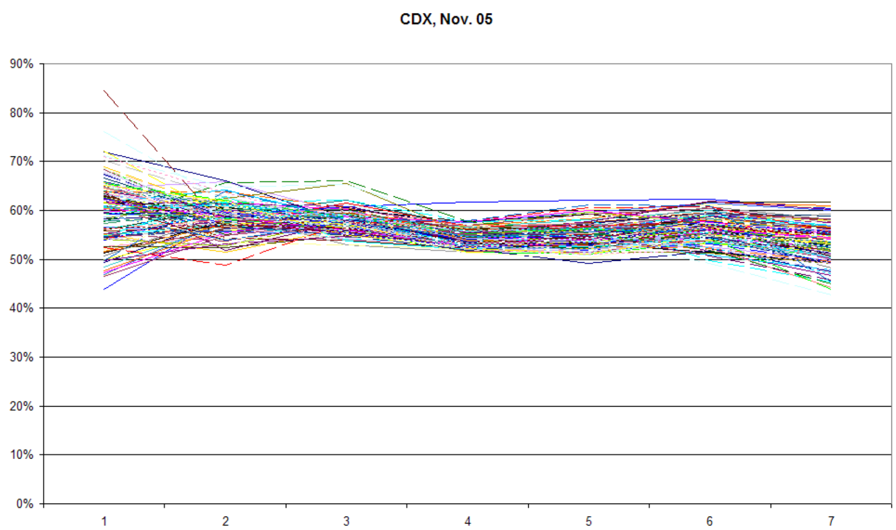

FIGURE 4. Term structure of equilibrium recovery swap rates for the CDX components, November 2005.

CLAUDIO ALBANESE, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, UNITED KINGDOM.
E-mail address: claudio.albanese@imperial.ac.uk

OLIVER CHEN, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE.
E-mail address: ochen@nus.edu.sg

ANTONIO DALESSANDRO, MOODY'S INVESTORS SERVICES, LONDON
E-mail address: antonio.dalessandro@imperial.ac.uk

ALICIA VIDLER, MERRILL LYNCH, LONDON
E-mail address: alicia_vidler@ml.com

CDX, Nov 05	1	2	3	4	5	7	10
0-1	1577.9	2584.8	3608.6	4273.9	4524.7	4466.8	4139.4
1-2	11.7	168.1	566.1	1240.4	1903.9	2342.0	2257.3
2-3	0.0	38.3	96.2	335.0	788.1	1526.8	1715.8
0-3	-0.40%	3.93%	12.38%	24.73%	38.67%	53.22%	52.98%
3-7	0.0	15.2	25.0	50.6	116.9	376.5	780.0
7-10	0.0	0.2	10.4	19.7	22.5	40.6	203.1
10-15	0.0	0.0	6.4	11.2	14.1	16.7	39.7
15-30	0.0	0.0	0.3	2.9	5.3	7.9	8.5
30-45	0.0	0.0	0.0	1.0	1.9	3.0	3.6
45-60		0.0	0.0	0.0	1.0	1.7	2.1
60-100		0.0	0.0	0.0	0.2	0.5	1.0

FIGURE 5. Term structure of model tranche spreads for November 2005. The 0-3 tranche is priced in terms of the upfront fee, while the running spread is 500 bp.

ITX, Nov 2005	1	2	3	4	5	7	10
0-1	1073.6	1762.6	2447.0	2947.6	3311.3	3491.5	3286.7
1-2	26.8	188.6	470.6	872.9	1313.2	1882.9	1968.6
2-3	0.3	38.0	95.4	248.5	503.8	1097.0	1456.7
3-6	0.0	12.7	23.8	47.2	93.4	319.1	764.0
0-3	-1.6%	1.0%	7.1%	16.0%	26.8%	43.6%	48.7%
6-9	0.0	0.9	10.5	17.9	20.9	41.1	201.7
9-12	0.0	0.0	6.1	10.2	13.0	16.2	44.1
12-16	0.0	0.0	1.7	4.8	7.9	10.8	14.0
16-22	0.0	0.0	0.1	2.6	4.4	7.1	7.6
22-30	0.0	0.0	0.0	1.5	2.4	4.2	4.7
30-40	0.0	0.0	0.0	0.3	1.2	2.1	2.8

FIGURE 6. Term structure of model tranche spreads for November 2005. The 0-3 tranche is priced in terms of the upfront fee, while the running spread is 500 bp.

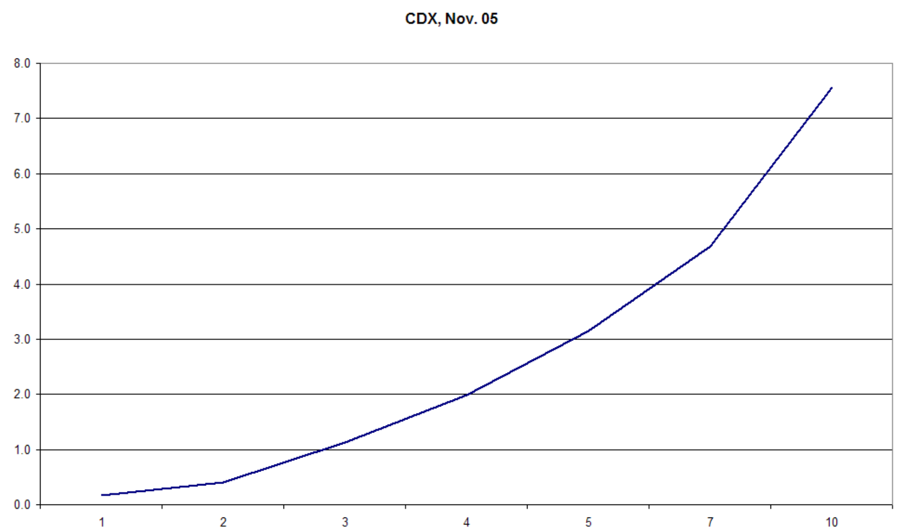

Figure 7. Model index basis for the CDX in Nov. 2005.

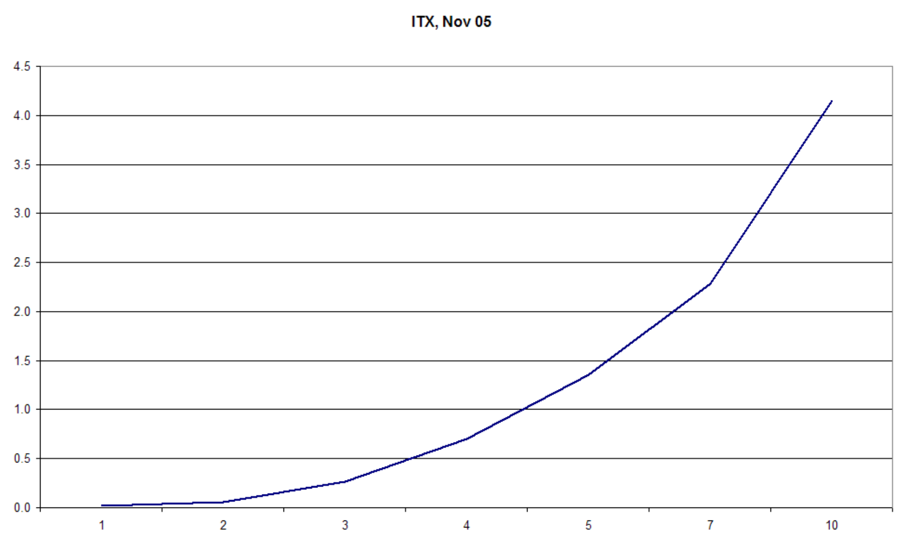

Figure 8. Model index basis for the iTraxx in Nov. 2005.

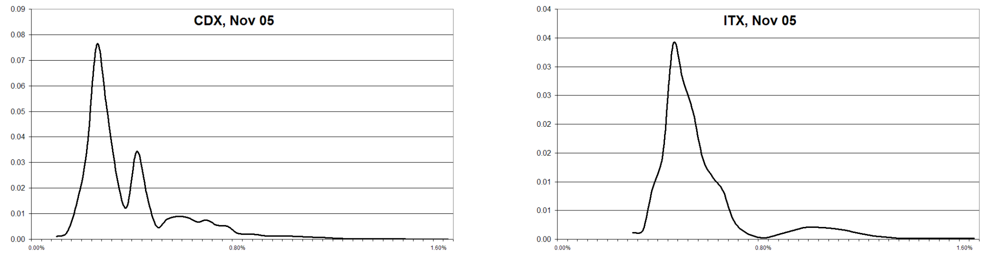

FIGURE 9. Implied 1 year loss distributions in November 2005.

.

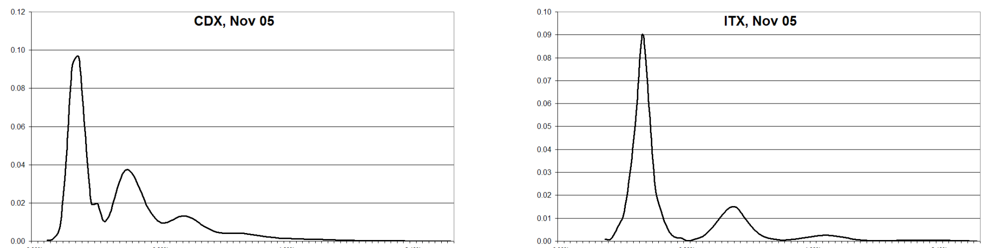

FIGURE 10. Implied 2 year loss distributions in November 2005.

.

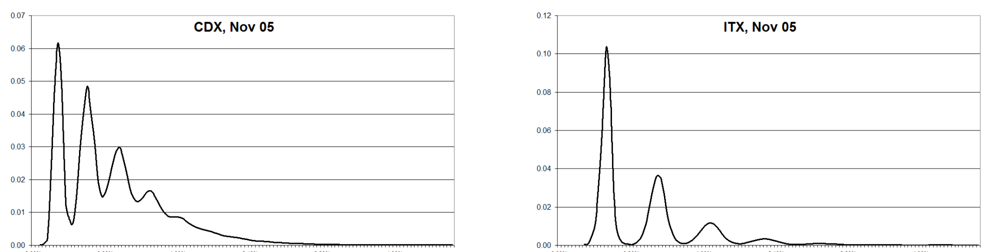

FIGURE 11. Implied 3 year loss distributions in November 2005.

.

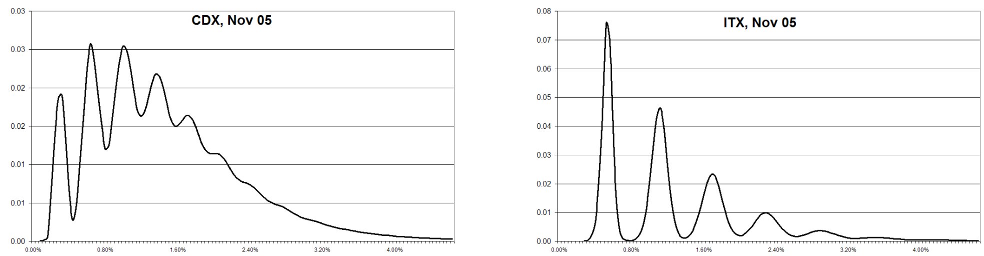

FIGURE 12. Implied 4 year loss distributions in November 2005.

.

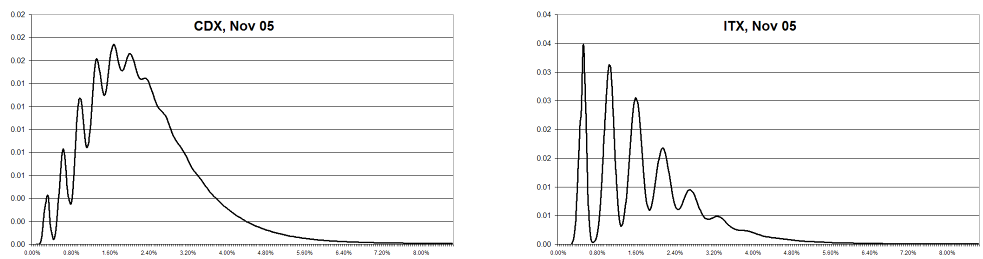

FIGURE 13. Implied 5 year loss distributions in November 2005.

.

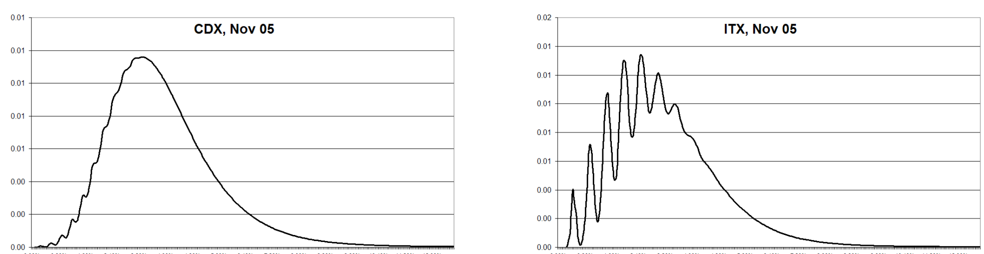

FIGURE 14. Implied 7 year loss distributions in November 2005.

.

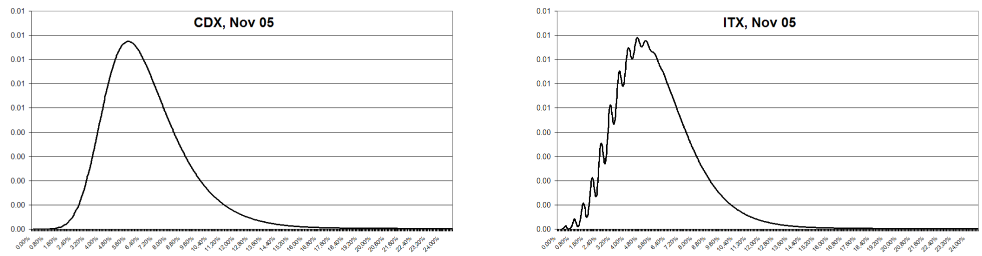

FIGURE 15. Implied 10 year loss distributions in November 2005.

.

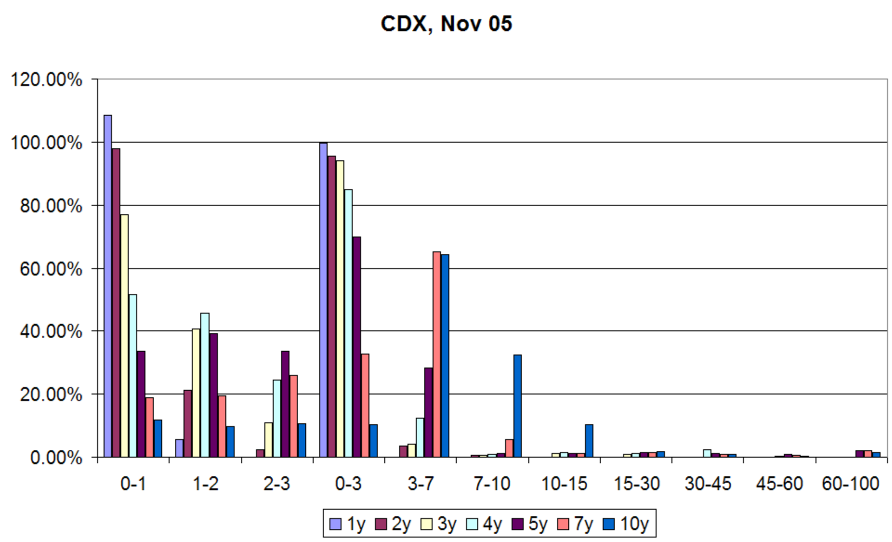

FIGURE 16. Index deltas for CDX tranches in Nov. 2005.

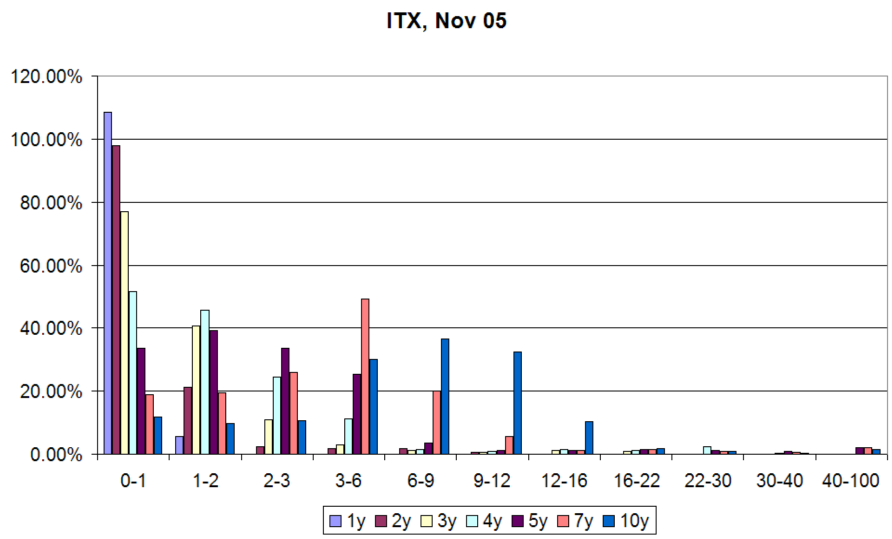

Figure 17. Index deltas for iTraxx tranches in Nov. 2005.

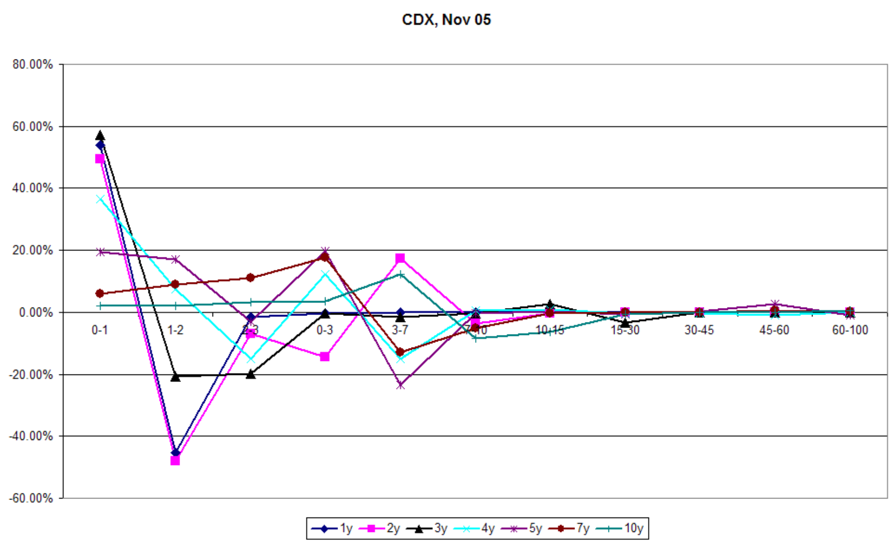

Figure 18. Index gammas for CDX tranches in Nov. 2005.

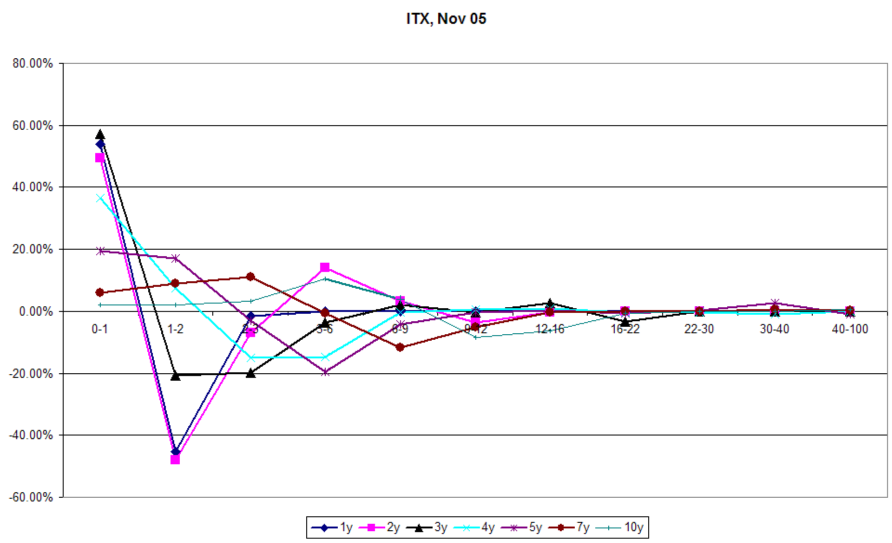

FIGURE 19. Index gammas for iTraxx tranches in Nov. 2005.

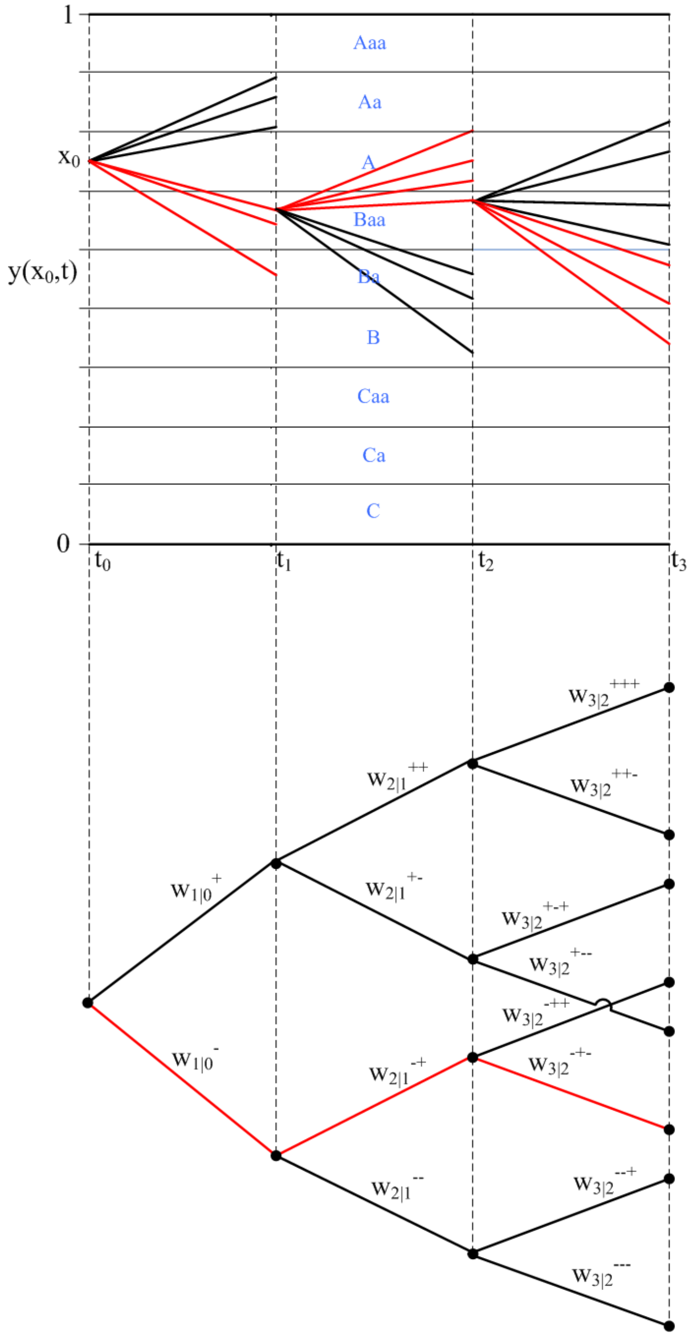

FIGURE 20. Tree for credit cycle scenarios versus conditional lattices.

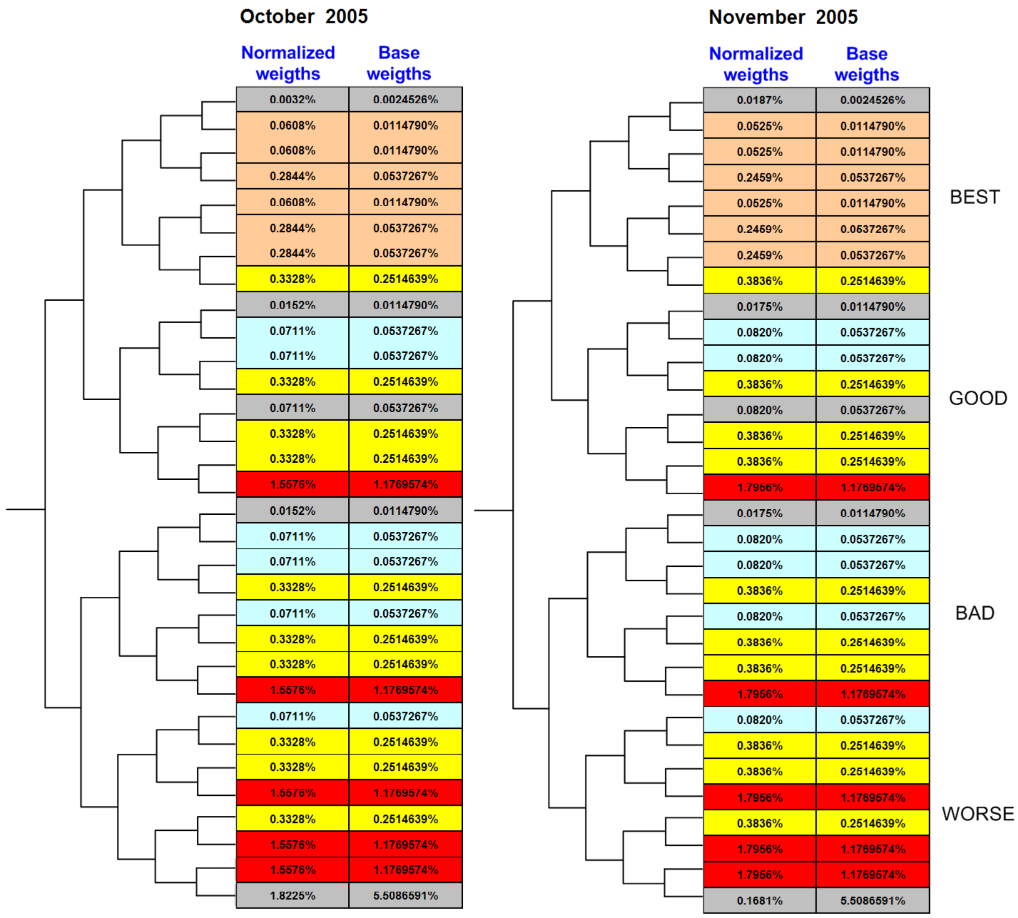

FIGURE 21. Specifications for the scenario weights on October 2005 and on November 2005 required to calibrate simultaneously the iTraxx and CDX indices. The different colors adopted indicate the scenarios that most relevantly affect the price.